Chapter 2

2.1

1. Not a linear transformation, since $y_2 = x_2 + 2$ is not linear in our sense.

3. Not linear, since $y_2 = x_1 x_3$ is nonlinear.

5. By Fact 2.1.2, the three columns of the 2×3 matrix A are $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$, so that

$$A = \begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{bmatrix}.$$

7. Note that $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = [\vec{v}_1 \ldots \vec{v}_m] \begin{bmatrix} x_1 \\ \cdots \\ x_m \end{bmatrix}$, so that T is indeed linear, with matrix $[\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_m]$.

9. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2. Reducing the system $\left| \begin{array}{ccccc} 2x_1 & + & 3x_2 & = & y_1 \\ 6x_1 & + & 9x_2 & = & y_2 \end{array} \right|$ we obtain $\left| \begin{array}{ll} x_1 + 1.5x_2 & = 0.5y_1 \\ 0 & = -3y_1 + y_2 \end{array} \right|$.

   No unique solution (x_1, x_2) can be found for a given (y_1, y_2); the matrix is noninvertible.

11. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2. Reducing the system $\left| \begin{array}{ccccc} x_1 & + & 2x_2 & = & y_1 \\ 3x_1 & + & 9x_2 & = & y_2 \end{array} \right|$ we find that $\left| \begin{array}{lcl} x_1 & = & 3y_1 - \frac{2}{3}y_2 \\ x_2 & = & -y_1 + \frac{1}{3}y_2 \end{array} \right|$. The inverse matrix is $\begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}$.

13. a. First suppose that $a \neq 0$. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2.

$$\left| \begin{array}{ccccc} ax_1 & + & bx_2 & = & y_1 \\ cx_1 & + & dx_2 & = & y_2 \end{array} \right| \div a \to \left| \begin{array}{ccccc} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ cx_1 & + & dx_2 & = & y_2 \end{array} \right| -c(I) \to$$

$$\left| \begin{array}{ccccccc} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & (d - \frac{bc}{a})x_2 & = & -\frac{c}{a}y_1 & + & y_2 \end{array} \right| \to$$

$$\left| \begin{array}{lllll} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ & & (\frac{ad-bc}{a})x_2 & = & -\frac{c}{a}y_1 + y_2 \end{array} \right|$$

We can solve this system for x_1 and x_2 if (and only if) $ad - bc \neq 0$, as claimed.

If $a = 0$, then we have to consider the system

$$\left| \begin{array}{lllll} & & bx_2 & = & y_1 \\ cx_1 & + & dx_2 & = & y_2 \end{array} \right| \text{swap} : I \leftrightarrow II \left| \begin{array}{lllll} cx_1 & + & dx_2 & = & y_2 \\ & & bx_2 & = & y_1 \end{array} \right|$$

We can solve for x_1 and x_2 provided that both b and c are nonzero, that is if $bc \neq 0$. Since $a = 0$, this means that $ad - bc \neq 0$, as claimed.

b. First suppose that $ad - bc \neq 0$ and $a \neq 0$. Let $D = ad - bc$ for simplicity. We continue our work in part (a):

$$\left| \begin{array}{lllll} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ & & \frac{D}{a}x_2 & = & -\frac{c}{a}y_1 + y_2 \end{array} \right| \cdot \frac{a}{D} \rightarrow$$

$$\left| \begin{array}{lllll} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ & & x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right| -\frac{b}{a}(II) \rightarrow$$

$$\left| \begin{array}{lll} x_1 & = & (\frac{1}{a} + \frac{bc}{aD})y_1 - \frac{b}{D}y_2 \\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right|$$

$$\left| \begin{array}{lll} x_1 & = & \frac{d}{D}y_1 - \frac{b}{D}y_2 \\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right|$$

(Note that $\frac{1}{a} + \frac{bc}{aD} = \frac{D+bc}{aD} = \frac{ad}{aD} = \frac{d}{D}$.)

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, as claimed. If $ad - bc \neq 0$ and $a = 0$, then we have to solve the system

$$\left| \begin{array}{lll} cx_1 + dx_2 & = & y_2 \\ bx_2 & = & y_1 \end{array} \right| \begin{array}{l} \div c \\ \div b \end{array}$$

$$\left| \begin{array}{lll} x_1 + \frac{d}{c}x_2 & = & \frac{1}{c}y_2 \\ x_2 & = & \frac{1}{b}y_1 \end{array} \right| -\frac{d}{c}(II)$$

$$\left| \begin{array}{lll} x_1 & = & -\frac{d}{bc}y_1 + \frac{1}{c}y_2 \\ x_2 & = & \frac{1}{b}y_1 \end{array} \right|$$

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (recall that $a = 0$), as claimed.

15. By Exercise 13a, the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is invertible if (and only if) $a^2 + b^2 \neq 0$, which is the case unless $a = b = 0$. If $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is invertible, then its inverse is $\frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, by Exercise 13b.

17. If $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A\vec{x} = -\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^2$, so that A represents a reflection about the origin.

    This transformation is its own inverse: $A^{-1} = A$.

19. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, so that A represents the orthogonal projection onto the $\vec{e}_1$ axis. (See Figure 2.1.) This transformation is not invertible, since the equation $A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has infinitely many solutions $\vec{x}$.

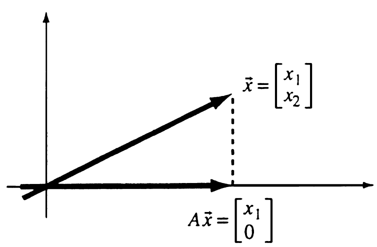

Figure 2.1: for Problem 2.1.19 .

21. Compare with Example 5.

    If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$. Note that the vectors $\vec{x}$ and $A\vec{x}$ are perpendicular and have the same length. If $\vec{x}$ is in the first quadrant, then $A\vec{x}$ is in the fourth. Therefore, A represents the rotation through an angle of 90° in the clockwise direction. (See Figure 2.2.) The inverse $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents the rotation through 90° in the counterclockwise direction.

23. Compare with Exercise 21.

    Note that $A = 2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so that A represents a rotation through an angle of 90° in the clockwise direction, followed by a scaling by the factor of 2.

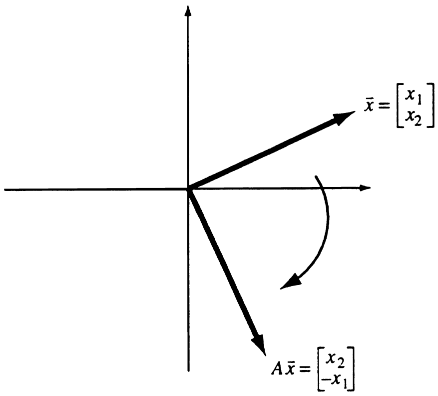

Figure 2.2: for Problem 2.1.21 .

The inverse $A^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ represents a rotation through an angle of 90° in the counterclockwise direction, followed by a scaling by the factor of $\frac{1}{2}$.

25. The matrix represents a scaling by the factor of 2. (See Figure 2.3.)

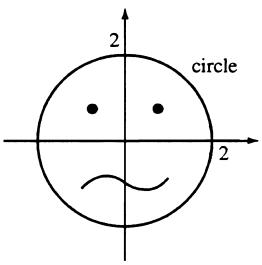

Figure 2.3: for Problem 2.1.25 .

27. This matrix represents a reflection about the $\vec{e}_1$ axis. (See Figure 2.4.)

29. This matrix represents a reflection about the origin. Compare with Exercise 17. (See Figure 2.5.)

31. The image must be reflected about the $\vec{e}_2$ axis, that is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ must be transformed into $\begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$: This can be accomplished by means of the linear transformation $T(\vec{x}) =$

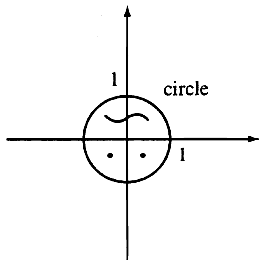

Figure 2.4: for Problem 2.1.27 .

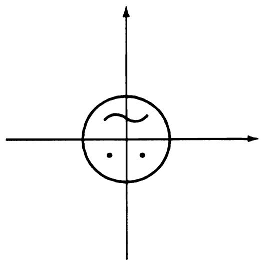

Figure 2.5: for Problem 2.1.29 .

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}.$$

33. By Fact 2.1.2, $A = \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$. (See Figure 2.6.)

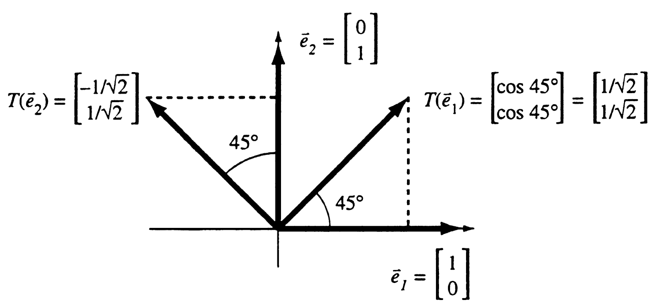

Figure 2.6: for Problem 2.1.33 .

Therefore, $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

35. We want to find a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $A\begin{bmatrix} 5 \\ 42 \end{bmatrix} = \begin{bmatrix} 89 \\ 52 \end{bmatrix}$ and $A\begin{bmatrix} 6 \\ 41 \end{bmatrix} = \begin{bmatrix} 88 \\ 53 \end{bmatrix}$.

This amounts to solving the system $\left| \begin{array}{llr} 5a + 42b & & = 89 \\ 6a + 41b & & = 88 \\ & 5c + 42d & = 52 \\ & 6c + 41d & = 53 \end{array} \right|$.

(Here we really have two systems with two unknowns each.)

The unique solution is $a = 1,\ b = 2,\ c = 2$, and $d = 1$, so that $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

37. Since $\vec{x} = \vec{v} + k(\vec{w} - \vec{v})$, we have $T(\vec{x}) = T\left(\vec{v} + k(\vec{w} - \vec{v})\right) = T(\vec{v}) + k(T(\vec{w}) - T(\vec{v}))$, by Fact 2.1.3

Since k is between 0 and 1, the tip of this vector $T(\vec{x})$ is on the line segment connecting the tips of $T(\vec{v})$ and $T(\vec{w})$. (See Figure 2.7.)

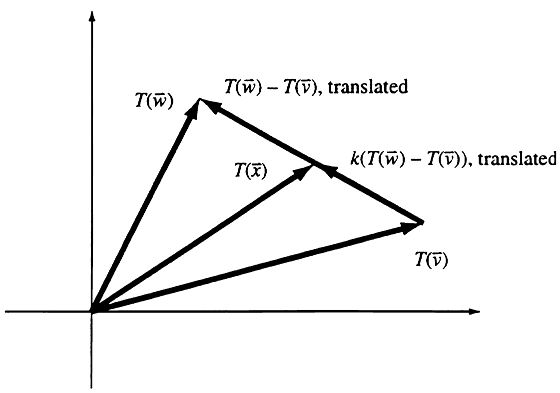

Figure 2.7: for Problem 2.1.37 .

39. By Fact 2.1.2, we have $T\begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & \dots & T(\vec{e}_m) \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = x_1 T(\vec{e}_1) + \dots + x_m T(\vec{e}_m)$.

41. These linear transformations are of the form $[y] = [a\ b]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, or $y = ax_1 + bx_2$. The graph of such a function is a plane through the origin.

43. a. $T(\vec{x}) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 3x_2 + 4x_3 = [2\ 3\ 4]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The transformation is indeed linear, with matrix $[2\ 3\ 4]$.

b. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then T is linear with matrix $[v_1\ v_2\ v_3]$, as in part (a).

c. Let $[a\ b\ c]$ be the matrix of T. Then $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [a\ b\ c]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ax_1 + bx_2 + cx_3 =$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ does the job.

45. Yes, $\vec{z} = L(T(\vec{x}))$ is also linear, which we will verify using Fact 2.1.3. Part a holds, since $L(T(\vec{v} + \vec{w})) = L(T(\vec{v}) + T(\vec{w})) = L(T(\vec{v})) + L(T(\vec{w}))$, and part b also works, because $L(T(k\vec{v})) = L(kT(\vec{v})) = kL(T(\vec{v}))$.

47. Write $\vec{w}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$: $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$. (See Figure 2.8.)

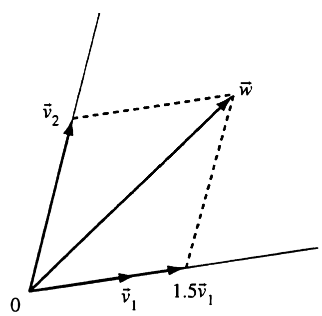

Figure 2.8: for Problem 2.1.47 .

Measurements show that we have *roughly* $\vec{w} = 1.5\vec{v}_1 + \vec{v}_2$.

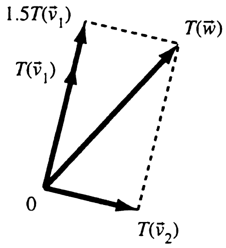

Figure 2.9: for Problem 2.1.47 .

Therefore, by linearity, $T(\vec{w}) = T(1.5\vec{v}_1 + \vec{v}_2) = 1.5T(\vec{v}_1) + T(\vec{v}_2)$. (See Figure 2.9.)

49. a. Let x_1 be the number of 2 Franc coins, and x_2 be the number of 5 Franc coins. Then $\left| \begin{matrix} 2x_1 & +5x_2 & = & 144 \\ x_1 & +x_2 & = & 51 \end{matrix} \right|$.

From this we easily find our solution vector to be $\begin{bmatrix} 37 \\ 14 \end{bmatrix}$.

b. $\begin{bmatrix} \text{total value of coins} \\ \text{total number of coins} \end{bmatrix} = \begin{bmatrix} 2x_1 & +5x_2 \\ x_1 & +x_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

So, $A = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}$.

c. By Exercise 13, matrix A is invertible (since $ad - bc = -3 \neq 0$), and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix}$.

Then $-\frac{1}{3}\begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 144 \\ 51 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 144 & -5(51) \\ -144 & +2(51) \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -111 \\ -42 \end{bmatrix} = \begin{bmatrix} 37 \\ 14 \end{bmatrix}$, which was the vector we found in part a.

51. a. $\begin{bmatrix} C \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9}(F - 32) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9}F - \frac{160}{9} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} F \\ 1 \end{bmatrix}$.

So $A = \begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix}$.

b. Using Exercise 13, we find $\frac{5}{9}(1) - (-\frac{160}{9})0 = \frac{5}{9} \neq 0$, so A is invertible.

$A^{-1} = \frac{9}{5}\begin{bmatrix} 1 & \frac{160}{9} \\ 0 & \frac{5}{9} \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & 32 \\ 0 & 1 \end{bmatrix}$. So, $F = \frac{9}{5}C + 32$.

53. First we notice that all entries along the diagonal must be 1, since those represent converting one currency to itself. Also, since $a_{34} = 200$, £1 = ¥200, so ¥1 = £$\frac{1}{200}$. So $a_{43} = \frac{1}{200}$. Using this same approach, we can find a_{21} and a_{41} as well.

    So far, $A = \begin{bmatrix} 1 & 0.8 & * & 1.5 \\ 1.25 & 1 & * & * \\ * & * & 1 & 200 \\ \frac{2}{3} & * & \frac{1}{200} & 1 \end{bmatrix}$.

    Now, using a_{43} and a_{14}, ¥1 = £$\frac{1}{200}$ and £1 = 1.5 Euros. So, ¥1 = $\frac{1}{200}$(1.5)Euros = $\frac{3}{400}$ Euros, meaning that $a_{13} = \frac{3}{400}$.

    We use this same approach to see that $a_{24} = a_{21}a_{14} = \frac{5}{4}(\frac{3}{2}) = \frac{15}{8}$, and $a_{23} = a_{21}a_{13} = \frac{5}{4}(\frac{3}{400}) = \frac{3}{320}$.

    Then, using our method from above to find a_{43}, we can find a_{31}, a_{42} and a_{32}.

    Thus, $A = \begin{bmatrix} 1 & \frac{4}{5} & \frac{3}{400} & \frac{3}{2} \\ \frac{5}{4} & 1 & \frac{3}{320} & \frac{15}{8} \\ \frac{400}{3} & \frac{320}{3} & 1 & 200 \\ \frac{2}{3} & \frac{8}{15} & \frac{1}{200} & 1 \end{bmatrix}$.

2.2

1. The standard L is transformed into a distorted L whose foot is the vector $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

   Meanwhile, the back becomes the vector $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

3. If $\vec{x}$ is in the unit square in $\mathbb{R}^2$, then $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$ with $0 \le x_1,\ x_2 \le 1$, so that

   $T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2)$.

   The image of the unit square is a parallelogram in $\mathbb{R}^3$; two of its sides are $T(\vec{e}_1)$ and $T(\vec{e}_2)$, and the origin is one of its vertices. (See Figure 2.10.)

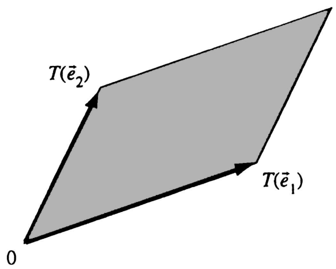

Figure 2.10: for Problem 2.2.3 .

5. Note that $\cos(\theta) = -0.8$, so that $\theta = \arccos(-0.8) \approx 2.498$.

7. According to the discussion on page 61, $\text{ref}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2\left(\vec{u} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)\vec{u} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, where $\vec{u}$ is a unit vector on L. To get $\vec{u}$, we normalize $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$:

$$\vec{u} = \tfrac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \text{ so that } \text{ref}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2(\tfrac{5}{3})\tfrac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{11}{9} \\ \frac{1}{9} \\ \frac{11}{9} \end{bmatrix}.$$

9. By Fact 2.2.5, this is a vertical shear.

11. In Exercise 10 we found the matrix $A = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}$ of the projection onto the line L. By Fact 2.2.2,

    $\text{ref}_L\vec{x} = 2(\text{proj}_L\vec{x}) - \vec{x} = 2A\vec{x} - \vec{x} = (2A - I_2)\vec{x}$, so that the matrix of the reflection is

    $2A - I_2 = \begin{bmatrix} 0.28 & 0.96 \\ 0.96 & -0.28 \end{bmatrix}$.

13. By Fact 2.2.2,

$$\text{ref}_L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= 2(u_1x_1 + u_2x_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (2u_1^2 - 1)x_1 + 2u_1u_2x_2 \\ 2u_1u_2x_1 + (2u_2^2 - 1)x_2 \end{bmatrix}.$$

The matrix is $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$. Note that the sum of the diagonal entries is $a + d = 2(u_1^2 + u_2^2) - 2 = 0$, since $\vec{u}$ is a unit vector. It follows that $d = -a$. Since $c = b$, A is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Also, $a^2 + b^2 = (2u_1^2 - 1)^2 + 4u_1^2u_2^2 = 4u_1^4 - 4u_1^2 + 1 + 4u_1^2(1 - u_1^2) = 1$, as claimed.

15. According to the discussion on Page 61, $\text{ref}_L(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$

$$= 2(x_1u_1 + x_2u_2 + x_3u_3) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1u_1^2 & +2x_2u_2u_1 & +2x_3u_3u_1 & -x_1 \\ 2x_1u_1u_2 & +2x_2u_2^2 & +2x_3u_3u_2 & -x_2 \\ 2x_1u_1u_3 & +2x_2u_2u_3 & +2x_3u_3^2 & -x_3 \end{bmatrix} = \begin{bmatrix} (2u_1^2 - 1)x_1 & +2u_2u_1x_2 & +2u_1u_3x_3 \\ 2u_1u_2x_1 & +(2u_2^2 - 1)x_2 & +2u_2u_3x_3 \\ 2u_1u_3x_1 & +2u_2u_3x_2 & +(2u_3^2 - 1)x_3 \end{bmatrix}.$$

So $A = \begin{bmatrix} (2u_1^2 - 1) & 2u_2u_1 & 2u_1u_3 \\ 2u_1u_2 & (2u_2^2 - 1) & 2u_2u_3 \\ 2u_1u_3 & 2u_2u_3 & (2u_3^2 - 1) \end{bmatrix}$.

17. We want, $\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 & +bv_2 \\ bv_1 & -av_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

Now, $(a-1)v_1 + bv_2 = 0$ and $bv_1 - (a+1)v_2$, which is a system with solutions of the form $\begin{bmatrix} bt \\ (1-a)t \end{bmatrix}$, where t is an arbitrary constant.

Let's choose $t = 1$, making $\vec{v} = \begin{bmatrix} b \\ 1 - a \end{bmatrix}$.

Similarly, we want $A\vec{w} = -\vec{w}$. We perform a computation as above to reveal $\vec{w} = \begin{bmatrix} a - 1 \\ b \end{bmatrix}$ as a possible choice. A quick check of $\vec{v} \cdot \vec{w} = 0$ reveals that they are indeed perpendicular.

Now, any vector $\vec{x}$ in $\mathbb{R}$ can be written in terms of components with respect to $L = \text{span}(\vec{v})$ as $\vec{x} = \vec{x}^{||} + \vec{x}^{\perp} = c\vec{v} + d\vec{w}$. Then, $T(\vec{x}) = A\vec{x} = A(c\vec{v} + d\vec{w}) = A(c\vec{v}) + A(d\vec{w}) = cA\vec{v} + dA\vec{w} = c\vec{v} - d\vec{w} = \vec{x}^{||} - \vec{x}^{\perp} = \text{ref}_L(\vec{x})$, by Definition 2.2.2.

(The vectors $\vec{v}$ and $\vec{w}$ constructed above are both zero in the special case that $a = 1$ and $b = 0$. In that case, we can let $\vec{v} = \vec{e}_1$ and $\vec{w} = \vec{e}_2$ instead.)

19. $T(\vec{e}_1) = \vec{e}_1$, $T(\vec{e}_2) = \vec{e}_2$, and $T(\vec{e}_3) = \vec{0}$, so that the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

21. $T(\vec{e}_1) = \vec{e}_2$, $T(\vec{e}_2) = -\vec{e}_1$, and $T(\vec{e}_3) = \vec{e}_3$, so that the matrix is $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (See Figure 2.11.)

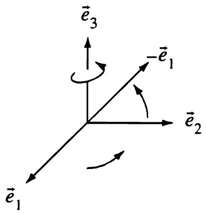

Figure 2.11: for Problem 2.2.21 .

23. $T(\vec{e}_1) = \vec{e}_3$, $T(\vec{e}_2) = \vec{e}_2$, and $T(\vec{e}_3) = \vec{e}_1$, so that the matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. (See Figure 2.12.)

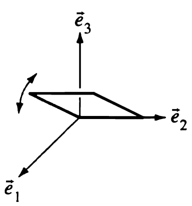

Figure 2.12: for Problem 2.2.23 .

25. The matrix $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ represents a horizontal shear, and its inverse $A^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ represents such a shear as well, but "the other way."

27. Matrix B clearly represents a scaling.

    Matrix C represents a projection, by Definition 2.2.1, with $u_1 = 0.6$ and $u_2 = 0.8$.

    Matrix E represents a shear, by Fact 2.2.5.

    Matrix A represents a reflection, by Definition 2.2.2.

    Matrix D represents a rotation, by Definition 2.2.3.

29. To check that L is linear, we verify the two parts of Fact 2.2.1.

a. Use the hint and apply L on both sides of the equation $\vec{x}+\vec{y}=T(L(\vec{x})+L(\vec{y}))$:

$L(\vec{x}+\vec{y})=L(T(L(\vec{x})+L(\vec{y})))=L(\vec{x})+L(\vec{y})$, as claimed.

b. $L(k\vec{x})=L(kT(L(\vec{x}))=L(T(kL(\vec{x})))=kL(\vec{x})$, as claimed.

$\uparrow$ $\uparrow$

$\vec{x}=T(L(\vec{x}))$ T is linear.

31. Write $A=[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$; then $A\vec{x}=[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_1\vec{v}_1+x_2\vec{v}_2+x_3\vec{v}_3$.

We must choose $\vec{v}_1,\vec{v}_2$, and $\vec{v}_3$ in such a way that $x_1\vec{v}_1+x_2\vec{v}_2+x_3\vec{v}_3$ is perpendicular to $\vec{w}=\begin{bmatrix}1\\2\\3\end{bmatrix}$ for all x_1,x_2, and x_3. This is the case if (and only if) all the vectors $\vec{v}_1,\vec{v}_2$, and $\vec{v}_3$ are perpendicular to $\vec{w}$, that is, if $\vec{v}_1\cdot\vec{w}=\vec{v}_2\cdot\vec{w}=\vec{v}_3\cdot\vec{w}=0$.

For example, we can choose $\vec{v}_1=\begin{bmatrix}-2\\1\\0\end{bmatrix}$ and $\vec{v}_2=\vec{v}_3=\vec{0}$, so that $A=\begin{bmatrix}-2&0&0\\1&0&0\\0&0&0\end{bmatrix}$.

33. Geometrically, we can find the representation $\vec{v}=\vec{v}_1+\vec{v}_2$ by means of a parallelogram, as shown in Figure 2.13.

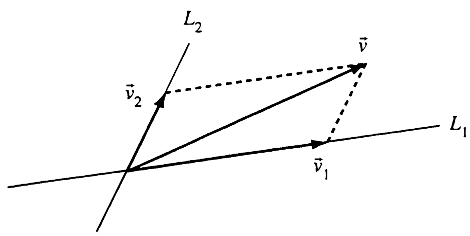

Figure 2.13: for Problem 2.2.33 .

To show the existence and uniqueness of this representation algebraically, choose a nonzero vector $\vec{w}_1$ in L_1 and a nonzero $\vec{w}_2$ in L_2. Then the system $x_1\vec{w}_1+x_2\vec{w}_2=\vec{0}$ or $[\vec{w}_1\ \vec{w}_2]\begin{bmatrix}x_1\\x_2\end{bmatrix}=\vec{0}$ has only the solution $x_1=x_2=0$ (if $x_1\vec{w}_1+x_2\vec{w}_2=\vec{0}$ then $x_1\vec{w}_1=-x_2\vec{w}_2$ is both in L_1 and in L_2, so that it must be the zero vector).

Therefore, the system $x_1\vec{w}_1 + x_2\vec{w}_2 = \vec{v}$ or $[\vec{w}_1 \ \vec{w}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{v}$ has a unique solution x_1, x_2 for all $\vec{v}$ in $\mathbb{R}^2$ (by Fact 1.3.4). Now set $\vec{v}_1 = x_1\vec{w}_1$ and $\vec{v}_2 = x_2\vec{w}_2$ to obtain the desired representation $\vec{v} = \vec{v}_1 + \vec{v}_2$. (Compare with Exercise 1.3.57.)

To show that the transformation $T(\vec{v}) = \vec{v}_1$ is linear, we will verify the two parts of Fact 2.1.3.

Let $\vec{v} = \underset{\text{in } L_1}{\underset{\uparrow}{\vec{v}_1}} + \underset{\text{in } L_2}{\underset{\uparrow}{\vec{v}_2}}$, $\vec{w} = \underset{\text{in } L_1}{\underset{\uparrow}{\vec{w}_1}} + \underset{\text{in } L_2}{\underset{\uparrow}{\vec{w}_2}}$, so that $\vec{v} + \vec{w} = \underset{\text{in } L_1}{\underset{\uparrow}{(\vec{v}_1 + \vec{w}_1)}} + \underset{\text{in } L_2}{\underset{\uparrow}{(\vec{v}_2 + \vec{w}_2)}}$ and $k\vec{v} = \underset{\text{in } L_1}{\underset{\uparrow}{k\vec{v}_1}} + \underset{\text{in } L_2}{\underset{\uparrow}{k\vec{v}_2}}$.

a. $T(\vec{v} + \vec{w}) = \vec{v}_1 + \vec{w}_1 = T(\vec{v}) + T(\vec{w})$, and

b. $T(k\vec{v}) = k\vec{v}_1 = kT(\vec{v})$, as claimed.

35. If the vectors $\vec{v}_1$ and $\vec{v}_2$ are defined as shown in Figure 2.14, then the parallelogram P consists of all vectors of the form $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$, where $0 \le c_1,\ c_2 \le 1$.

The image of P consists of all vectors of the form $T(\vec{v}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$.

These vectors form the parallelogram shown in Figure 2.1.14.

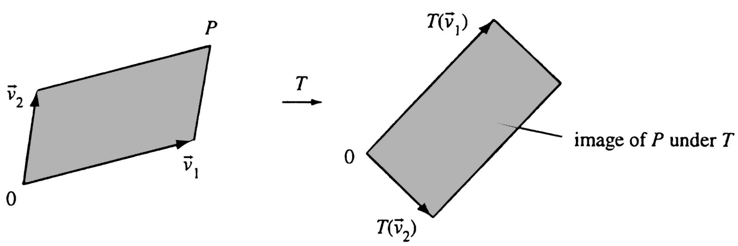

Figure 2.14: for Problem 2.2.35 .

37. a. By Definition 2.2.1, a projection has a matrix of the form $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$, where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector.

So the trace is $u_1^2 + u_2^2 = 1$.

b. By Definition 2.2.2, reflection matrices look like $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, so the trace is $a - a = 0$.

c. According to Fact 2.2.3, a rotation matrix has the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, so the trace is $\cos\theta + \cos\theta = 2\cos\theta$ for some θ. Thus, the trace is in the interval $[-2, 2]$.

d. By Fact 2.2.5, the matrix of a shear appears as either $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, depending on whether it represents a vertical or horizontal shear. In both cases, however, the trace is $1 + 1 = 2$.

39. a. Note that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. The matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ represents an orthogonal projection (Definition 2.2.1),

with $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$. So, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ represents a projection combined with a scaling by a factor of 2.

b. This looks similar to a shear, with the one zero off the diagonal. Since the two diagonal entries are identical, we can write $\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}$, showing that this matrix represents a vertical shear combined with a scaling by a factor of 3.

c. We are asked to write $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = k\begin{bmatrix} \frac{3}{k} & \frac{4}{k} \\ \frac{4}{k} & -\frac{3}{k} \end{bmatrix}$, with our scaling factor k yet to be determined. This matrix, $\begin{bmatrix} \frac{3}{k} & \frac{4}{k} \\ \frac{4}{k} & -\frac{3}{k} \end{bmatrix}$ has the form of a reflection matrix $\left(\begin{bmatrix} a & b \\ b & -a \end{bmatrix}\right)$. This form further requires that $1 = a^2 + b^2 = (\frac{3}{k})^2 + (\frac{4}{k})^2$, or $k = 5$. Thus, the matrix represents a reflection combined with a scaling by a factor of 5.

41. $\text{ref}_Q\vec{x} = -\text{ref}_P\vec{x}$ since $\text{ref}_Q\vec{x}$, $\text{ref}_P\vec{x}$, and $\vec{x}$ all have the same length, and $\text{ref}_Q\vec{x}$ and $\text{ref}_P\vec{x}$ enclose an angle of $2\alpha + 2\beta = 2(\alpha + \beta) = \pi$. (See Figure 2.15.)

43. Since $\vec{y} = A\vec{x}$ is obtained from $\vec{x}$ by a rotation through θ in the counterclockwise direction, $\vec{x}$ is obtained from $\vec{y}$ by a rotation through θ in the *clockwise* direction, that is, a rotation through $-\theta$. (See Figure 2.16.)

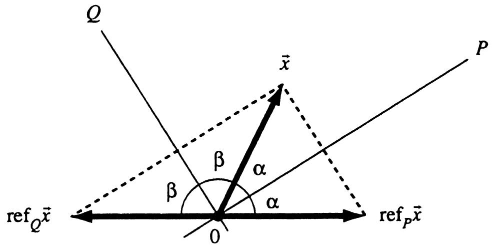

Figure 2.15: for Problem 2.2.41 .

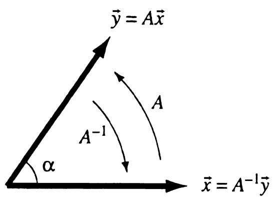

Figure 2.16: for Problem 2.2.43 .

Therefore, the matrix of the inverse transformation is $A^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. You can use the formula in Exercise 2.1.13b to check this result.

45. By Exercise 2.1.13, $A^{-1} = \frac{1}{-a^2-b^2}\begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \frac{1}{-(a^2+b^2)}\begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = -1\begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$.

So $A^{-1} = A$, which makes sense. Reflecting a vector twice about the same line will return it to its original state.

47. Write $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$.

a. $f(t) = \left(T\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}\right) \cdot \left(T\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}\right) = \begin{bmatrix} a\cos t + b\sin t \\ c\cos t + d\sin t \end{bmatrix} \cdot \begin{bmatrix} -a\sin t + b\cos t \\ -c\sin t + d\cos t \end{bmatrix}$

$= (a\cos t + b\sin t)(-a\sin t + b\cos t) + (c\cos t + d\sin t)(-c\sin t + d\cos t)$

This function $f(t)$ is continuous, since $\cos(t)$, $\sin(t)$, and constant functions are continuous, and sums and products of continuous functions are continuous.

b. $f\left(\frac{\pi}{2}\right) = T\begin{bmatrix}0\\1\end{bmatrix} \cdot T\begin{bmatrix}-1\\0\end{bmatrix} = -\left(T\begin{bmatrix}0\\1\end{bmatrix} \cdot T\begin{bmatrix}1\\0\end{bmatrix}\right)$, since T is linear.

$f(0) = T\begin{bmatrix}1\\0\end{bmatrix} \cdot T\begin{bmatrix}0\\1\end{bmatrix} = T\begin{bmatrix}0\\1\end{bmatrix} \cdot T\begin{bmatrix}1\\0\end{bmatrix}$. The claim follows.

c. By part (b), the numbers $f(0)$ and $f\left(\frac{\pi}{2}\right)$ have different signs (one is positive and the other negative), or they are both zero. Since $f(t)$ is continuous, by part (a), we can apply the intermediate value theorem. (See Figure 2.17.)

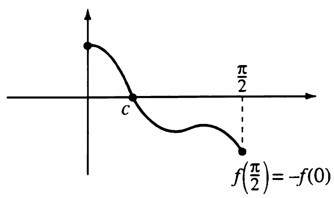

Figure 2.17: for Problem 2.2.47c .

d. Note that $\begin{bmatrix}\cos(t)\\\sin(t)\end{bmatrix}$ and $\begin{bmatrix}-\sin(t)\\\cos(t)\end{bmatrix}$ are perpendicular unit vectors, for any t. If we set

$\vec{v}_1 = \begin{bmatrix}\cos(c)\\\sin(c)\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}-\sin(c)\\\cos(c)\end{bmatrix}$, with the number c we found in part (c), then $f(c) = T(\vec{v}_1) \cdot T(\vec{v}_2) = 0$, so that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are perpendicular, as claimed. Note that $T(\vec{v}_1)$ or $T(\vec{v}_2)$ may be zero.

49. If $\vec{x} = \begin{bmatrix}\cos(t)\\\sin(t)\end{bmatrix}$ then $T(\vec{x}) = \begin{bmatrix}5 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}\cos(t)\\\sin(t)\end{bmatrix} = \begin{bmatrix}5\cos(t)\\2\sin(t)\end{bmatrix} = \cos(t)\begin{bmatrix}5\\0\end{bmatrix} + \sin(t)\begin{bmatrix}0\\2\end{bmatrix}$.

These vectors form an ellipse; consider the characterization of an ellipse given in the footnote on page 70, with $\vec{w}_1 = \begin{bmatrix}5\\0\end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix}0\\2\end{bmatrix}$. (See Figure 2.18.)

51. Consider the linear transformation T with matrix $A = [\vec{w}_1 \quad \vec{w}_2]$, that is,

$$T\begin{bmatrix}x_1\\x_2\end{bmatrix} = A\begin{bmatrix}x_1\\x_2\end{bmatrix} = [\vec{w}_1 \quad \vec{w}_2]\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\vec{w}_1 + x_2\vec{w}_2.$$

The curve C is the image of the unit circle under the transformation T: if $\vec{v} = \begin{bmatrix}\cos(t)\\\sin(t)\end{bmatrix}$ is on the unit circle, then $T(\vec{v}) = \cos(t)\vec{w}_1 + \sin(t)\vec{w}_2$ is on the curve C. Therefore, C is an ellipse, by Exercise 50. (See Figure 2.19.)

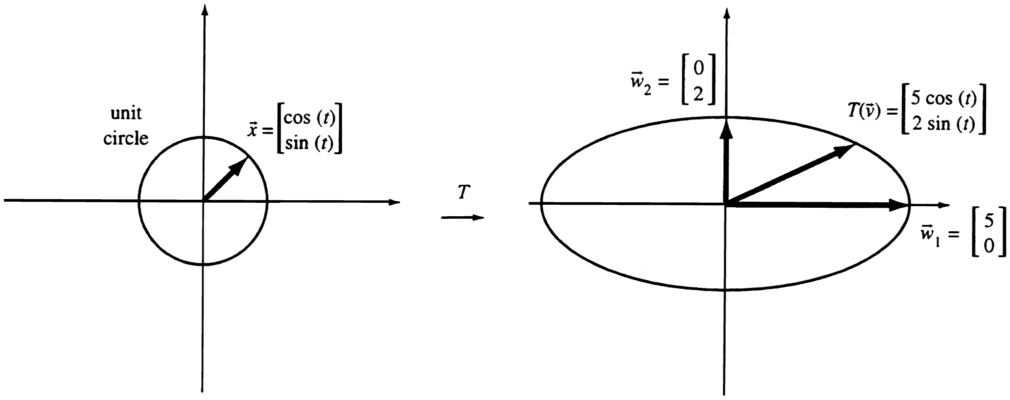

Figure 2.18: for Problem 2.2.49 .

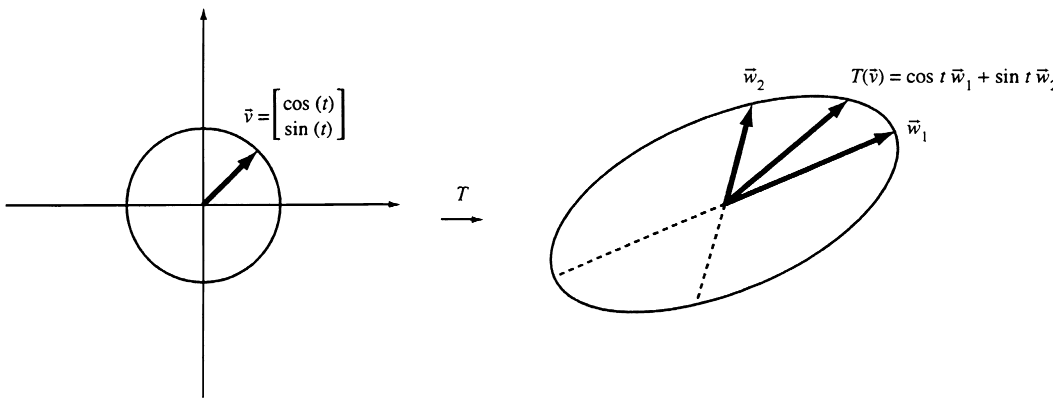

Figure 2.19: for Problem 2.2.51 .

2.3

1. rref $\begin{bmatrix} 2 & 3 & \vdots & 1 & 0 \\ 5 & 8 & \vdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & 8 & -3 \\ 0 & 1 & \vdots & -5 & 2 \end{bmatrix}$, so that $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$.

3. rref $\begin{bmatrix} 0 & 2 & \vdots & 1 & 0 \\ 1 & 1 & \vdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & -\frac{1}{2} & 1 \\ 0 & 1 & \vdots & \frac{1}{2} & 0 \end{bmatrix}$, so that $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

5. rref $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so that the matrix fails to be invertible, by Fact 2.3.3.

7. rref $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so that the matrix fails to be invertible, by Fact 2.3.3.

9. rref $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so that the matrix fails to be invertible, by Fact 2.3.3.

11. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

13. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$.

15. Use Fact 2.3.5; the inverse is $\begin{bmatrix} -6 & 9 & -5 & 1 \\ 9 & -1 & -5 & 2 \\ -5 & -5 & 9 & -3 \\ 1 & 2 & -3 & 1 \end{bmatrix}$

17. We make an attempt to solve for x_1 and x_2 in terms of y_1 and y_2:

$$\left| \begin{array}{lcl} x_1 + 2x_2 & = & y_1 \\ 4x_1 + 8x_2 & = & y_2 \end{array} \right| \xrightarrow[-4(I)]{} \left| \begin{array}{lcl} x_1 + 2x_2 & = & y_1 \\ 0 & = & -4y_1 + y_2 \end{array} \right|.$$

This system has no solutions (x_1, x_2) for some (y_1, y_2), and infinitely many solutions for others; the transformation fails to be invertible.

19. Solving for x_1, x_2, and x_3 in terms of y_1, y_2, and y_3, we find that

$$\begin{aligned} x_1 &= 3y_1 - \tfrac{5}{2}y_2 + \tfrac{1}{2}y_3 \\ x_2 &= -3y_1 + 4y_2 - y_3 \\ x_3 &= y_1 - \tfrac{3}{2}y_2 + \tfrac{1}{2}y_3 \end{aligned}$$

21. $f(x) = x^2$ fails to be invertible, since the equation $f(x) = x^2 = 1$ has two solutions, $x = \pm 1$.

23. Note that $f'(x) = 3x^2 + 1$ is always positive; this implies that the function $f(x) = x^3 + x$ is increasing throughout. Therefore, the equation $f(x) = b$ has *at most* one solution x for all b. (See Figure 2.20.)

Now observe that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$; this implies that the equation $f(x) = b$ has at least one solution x for a given b (for a careful proof, use the intermediate value theorem; compare with Exercise 2.2.47c).

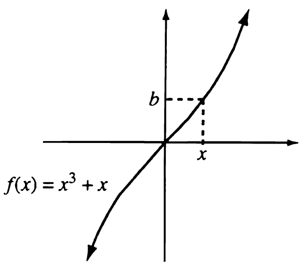

Figure 2.20: for Problem 2.3.23 .

25. Invertible, with inverse $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt[3]{y_1} \\ y_2 \end{bmatrix}$

27. This fails to be invertible, since the equation $\begin{bmatrix} x_1 + x_2 \\ x_1 x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solution.

29. Use Fact 2.3.3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix} \begin{matrix} \\ -I \\ -I \end{matrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 3 & k^2-1 \end{bmatrix} \begin{matrix} -II \\ \\ -3(II) \end{matrix} \to \begin{bmatrix} 1 & 0 & 2-k \\ 0 & 1 & k-1 \\ 0 & 0 & k^2-3k+2 \end{bmatrix}$$

The matrix is invertible if (and only if) $k^2 - 3k + 2 = (k-2)(k-1) \neq 0$, in which case we can further reduce it to I_3. Therefore, the matrix is invertible if $k \neq 1$ and $k \neq 2$.

31. Use Fact 2.3.3; first assume that $a \neq 0$.

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \begin{matrix} \text{swap}: \\ I \leftrightarrow II \end{matrix} \to \begin{bmatrix} -a & 0 & c \\ 0 & a & b \\ -b & -c & 0 \end{bmatrix} \begin{matrix} \div(-a) \\ \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & a & b \\ -b & -c & 0 \end{bmatrix} \begin{matrix} \\ \\ +b(I) \end{matrix} \to \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & a & b \\ 0 & -c & -\frac{bc}{a} \end{bmatrix} \begin{matrix} \\ \div a \\ \\ \end{matrix} \to$$

$$\begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & 1 & \frac{b}{a} \\ 0 & -c & -\frac{bc}{a} \end{bmatrix} \begin{matrix} \\ \\ +c(II) \end{matrix} \to \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & 0 \end{bmatrix}$$

Now consider the case when $a = 0$:

$\begin{bmatrix} 0 & 0 & b \\ 0 & 0 & c \\ -b & -c & 0 \end{bmatrix} \begin{matrix} \text{swap :} \\ I \leftrightarrow III \end{matrix} \rightarrow \begin{bmatrix} -b & -c & 0 \\ 0 & 0 & c \\ 0 & 0 & b \end{bmatrix}$: The second entry on the diagonal of rref will be 0.

It follows that the matrix $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ is noninvertible, regardless of the values of $a, b,$ and c.

33. Use Fact 2.3.6.

The requirement $A^{-1} = A$ means that $-\frac{1}{a^2+b^2}\begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. This is the case if (and only if) $a^2 + b^2 = 1$.

35. a. A is invertible if (and only if) all its diagonal entries, $a, d,$ and f, are nonzero.

b. As in part (a): if all the diagonal entries are nonzero.

c. Yes, A^{-1} will be upper triangular as well; as you construct rref$[A \vdots I_n]$, you will perform only the following row operations:

- divide rows by scalars
- subtract a multiple of the jth row from the ith row, where $j > i$.

Applying these operations to I_n, you end up with an upper triangular matrix.

d. As in part (b): if all diagonal entries are nonzero.

37. Make an attempt to solve the linear transformation $\vec{y} = (cA)\vec{x} = c(A\vec{x})$ for $\vec{x}$:

$A\vec{x} = \frac{1}{c}\vec{y}$, so that $\vec{x} = A^{-1}\left(\frac{1}{c}\vec{y}\right) = \left(\frac{1}{c}A^{-1}\right)\vec{y}$.

This shows that cA is indeed invertible, with $(cA)^{-1} = \frac{1}{c}A^{-1}$.

39. Suppose the ijth entry of M is k, and all other entries are as in the identity matrix. Then we can find rref$[M \vdots I_n]$ by subtracting k times the jth row from the ith row. Therefore, M is indeed invertible, and M^{-1} differs from the identity matrix only at the ijth entry; that entry is $-k$. (See Figure 2.21.)

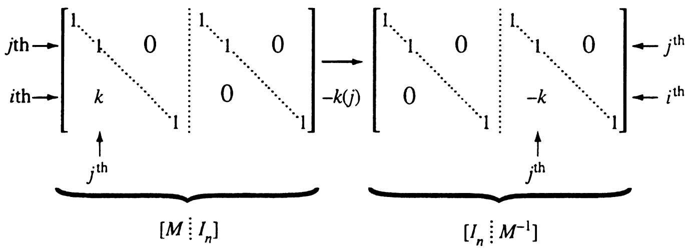

Figure 2.21: for Problem 2.3.39 .

41. a. Invertible: the transformation is its own inverse.

    b. Not invertible: the equation $T(\vec{x}) = \vec{b}$ has infinitely many solutions if $\vec{b}$ is on the plane, and none otherwise.

    c. Invertible: The inverse is a scaling by $\frac{1}{5}$ (that is, a contraction by 5). If $\vec{y} = 5\vec{x}$, then $\vec{x} = \frac{1}{5}\vec{y}$.

    d. Invertible: The inverse is a rotation about the same axis through the same angle in the opposite direction.

43. We make an attempt to solve the equation $\vec{y} = A(B\vec{x})$ for $\vec{x}$:

    $B\vec{x} = A^{-1}\vec{y}$, so that $\vec{x} = B^{-1}(A^{-1}\vec{y})$.

45. a. Each of the three row divisions requires three multiplicative operations, and each of the six row subtractions requires three multiplicative operations as well; altogether, we have $3 \cdot 3 + 6 \cdot 3 = 9 \cdot 3 = 3^3 = 27$ operations.

    b. Suppose we have already taken care of the first m columns: $[A \vdots I_n]$ has been reduced the matrix in Figure 2.22.

    Here, the stars represent arbitrary entries.

    Suppose the $(m+1)$th entry on the diagonal is k. Dividing the $(m+1)$th row by k requires n operations: $n-m-1$ to the left of the dotted line $\left(\text{not counting the computation } \frac{k}{k} = 1\right)$, and $m+1$ to the right of the dotted line $\left(\text{including } \frac{1}{k}\right)$. Now the matrix has the form showing in Figure 2.23.

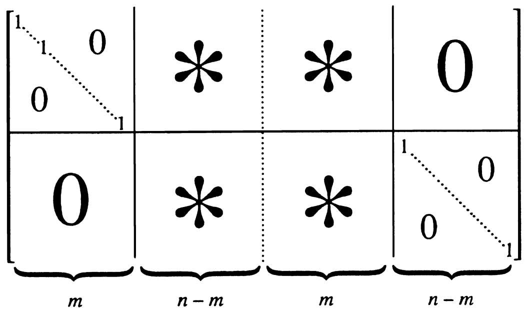

Figure 2.22: for Problem 2.3.45b .

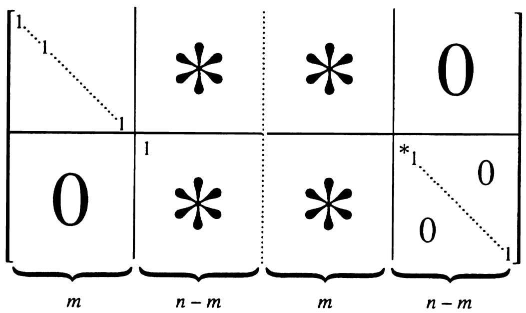

Figure 2.23: for Problem 2.3.45b .

Eliminating each of the other $n-1$ components of the $(m+1)$th column now requires n multiplicative operations ($n-m-1$ to the left of the dotted line, and $m+1$ to the right). Altogether, it requires $n+(n-1)n=n^2$ operations to process the mth column. To process all n columns requires $n \cdot n^2 = n^3$ operations.

c. The inversion of a 12×12 matrix requires $12^3 = 4^3 3^3 = 64 \cdot 3^3$ operations, that is, 64 times as much as the inversion of a 3×3 matrix. If the inversion of a 3×3 matrix takes one second, then the inversion of a 12×12 matrix takes 64 seconds.

47. Let $f(x) = x^2$; the equation $f(x) = 0$ has the unique solution $x = 0$.

49. a. $A = \begin{bmatrix} 0.293 & 0 & 0 \\ 0.014 & 0.207 & 0.017 \\ 0.044 & 0.01 & 0.216 \end{bmatrix}$, $I_3 - A = \begin{bmatrix} 0.707 & 0 & 0 \\ -0.014 & 0.793 & -0.017 \\ -0.044 & -0.01 & 0.784 \end{bmatrix}$

$$(I_3 - A)^{-1} = \begin{bmatrix} 1.41 & 0 & 0 \\ 0.0267 & 1.26 & 0.0274 \\ 0.0797 & 0.0161 & 1.28 \end{bmatrix}$$

b. We have $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so that $\vec{x} = (I_3 - A)^{-1}\vec{e}_1 =$ first column of $(I_3 - A)^{-1} \approx \begin{bmatrix} 1.41 \\ 0.0267 \\ 0.0797 \end{bmatrix}$.

c. As illustrated in part (b), the ith column of $(I_3 - A)^{-1}$ gives the output vector required to satisfy a consumer demand of 1 unit on industry i, in the absence of any other consumer demands. In particular, the ith diagonal entry of $(I_3 - A)^{-1}$ gives the output of industry i required to satisfy this demand. Since industry i has to satisfy the consumer demand of 1 as well as the interindustry demand, its total output will be at least 1.

d. Suppose the consumer demand increases from $\vec{b}$ to $\vec{b} + \vec{e}_2$ (that is, the demand on manufacturing increases by one unit). Then the output must change from $(I_3 - A)^{-1}\vec{b}$ to

$(I_3 - A)^{-1}(\vec{v} + \vec{e}_2) = (I_3 - A)^{-1}\vec{b} + (I_3 - A)^{-1}\vec{e}_2 = (I_3 - A)^{-1}\vec{b} +$ (second column of $(I_3 - A)^{-1}$).

The components of the second column of $(I_3 - A)^{-1}$ tells us by how much each industry has to increase its output.

e. The ijth entry of $(I_n - A)^{-1}$ gives the required increase of the output x_i of industry i to satisfy an increase of the consumer demand b_j on industry j by one unit. In the language of multivariable calculus, this quantity is $\frac{\partial x_i}{\partial b_j}$.

51. a. Since rank$(A) < n$, the matrix $E =$rref(A) will not have a leading one in the last row, and all entries in the last row of E will be zero.

Let $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. Then the last equation of the system $E\vec{x} = \vec{c}$ reads $0 = 1$, so this system is inconsistent.

Now, we can "rebuild" $\vec{b}$ from $\vec{c}$ by performing the reverse row-operations in the opposite order on $\left[E \vdots \vec{c}\right]$ until we reach $\left[A \vdots \vec{b}\right]$. Since $E\vec{x} = \vec{c}$ is inconsistent, $A\vec{x} = \vec{b}$ is inconsistent as well.

b. Since rank$(A) \leq \min(n, m)$, and $m < n$, rank$(A) < n$ also. Thus, by part a, there is a $\vec{b}$ such that $A\vec{x} = \vec{b}$ is inconsistent.

53. a. $A - \lambda I_2 = \begin{bmatrix} 3-\lambda & 1 \\ 3 & 5-\lambda \end{bmatrix}$.

This fails to be invertible when $(3-\lambda)(5-\lambda)-3=0$,

or $15-8\lambda+\lambda^2-3=0$,

or $12-8\lambda+\lambda^2=0$

or $(6-\lambda)(2-\lambda)=0$. So $\lambda=6$ or $\lambda=2$.

b. For $\lambda=6$, $A-\lambda I_2 = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$.

The system $(A-6I_2)\vec{x}=\vec{0}$ has the solutions $\begin{bmatrix} t \\ 3t \end{bmatrix}$, where t is an arbitrary constant.

Pick $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, for example.

For $\lambda=2$, $A-\lambda I_2 = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$.

The system $(A-2I_2)\vec{x}=\vec{0}$ has the solutions $\begin{bmatrix} t \\ -t \end{bmatrix}$, where t is an arbitrary constant.

Pick $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, for example.

c. For $\lambda=6$, $A\vec{x} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda=2$, $A\vec{x} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

2.4

1. $\begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$

3. Undefined

5. $\begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

11. $[10]$

13. $[h]$

15. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; Fact 2.4.9 applies to square matrices only.

17. Not necessarily true; $(A+B)^2 = (A+B)(A+B) = A^2+AB+BA+B^2 \neq A^2+2AB+B^2$ if $AB \neq BA$.

19. Not necessarily true; consider the case $A = I_n$ and $B = -I_n$.

21. True; $ABB^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$.

23. True; $(ABA^{-1})^3 = ABA^{-1}ABA^{-1}ABA^{-1} = AB^3A^{-1}$.

25. True; $(A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A$ (use Fact 2.4.8).

27. $\left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \end{bmatrix}[3] & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \end{bmatrix}[4] \\ \hline \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}+[4][3] & \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}+[4][4] \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline [19] & [16] \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 19 & 16 \end{bmatrix}$

29. The columns of B must be solutions of the system $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

    The solutions are of the form $B = \begin{bmatrix} -3t & -3s \\ t & s \end{bmatrix}$, where t and s are arbitrary constants, with at least one of them being nonzero.

31. The two columns of A must be solutions of the linear systems $B\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $B\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. Each of these systems has *infinitely many solutions.*

    The solutions are of the form $\begin{bmatrix} 2+t & -1+s \\ -1-2t & 1-2s \\ t & s \end{bmatrix}$.

33. By Fact 1.3.3, there is a nonzero $\vec{x}$ such that $B\vec{x} = \vec{0}$ and therefore $AB\vec{x} = \vec{0}$. By Fact 2.3.4b, the 3×3 matrix AB fails to be invertible.

35. a. Consider a solution $\vec{x}$ of the equation $A\vec{x} = \vec{0}$.

Multiply both sides by B from the left: $BA\vec{x} = B\vec{0} = \vec{0}$, so that $\vec{x} = \vec{0}$ (since $BA = I_m$).

It follows that $\vec{x} = \vec{0}$ is the only solution of $A\vec{x} = \vec{0}$.

b. $\vec{x} = A\vec{b}$ is a solution, since $B\vec{x} = BA\vec{b} = \vec{b}$ (because $BA = I_m$).

c. rank$(A) = m$, by part (a) (all variables are leading).

rank$(B) = m$, by part (b) (compare with Exercise 2.3.51a).

d. $m = \text{rank}(B) \leq$ (number of columns of B) $= n$

37. We want $S^{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S = S\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

So $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}$.

Thus, $c = a$ and $d = -b$. Matrix S must be of the form $\begin{bmatrix} a & b \\ a & -b \end{bmatrix}$ where $-ab - ab \neq 0$, or $-2ab \neq 0$, or $a \neq 0$ and $b \neq 0$.

39. Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we want $X\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}X$, or $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, or $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, meaning that $b = c = 0$. Also, we want $X\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}X$, or $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, or $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$ so $a = d$. Thus, $X = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI_2$ must be a multiple of the identity matrix. (X will then commute with any 2×2 matrix M, since $XM = aM = MX$.)

41. a. $D_\alpha D_\beta$ and $D_\beta D_\alpha$ are the same transformation, namely, a rotation through $\alpha + \beta$.

b. $$D_\alpha D_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$D_\beta D_\alpha$ yields the same answer.

43. Let A represent the rotation through 120°; then A^3 represents the rotation through 360°, that is $A^3 = I_2$.

$$A = \begin{bmatrix} \cos(120^\circ) & -\sin(120^\circ) \\ \sin(120^\circ) & \cos(120^\circ) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

45. We want A such that $A\vec{v}_i = \vec{w}_i$, for $i = 1, 2, \ldots, m$, or $A[\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_m] = [\vec{w}_1\ \vec{w}_2\ \ldots\ \vec{w}_m]$, or $AS = B$.

Multiplying by S^{-1} from the right we find the unique solution $A = BS^{-1}$.

47. Use the result of Exercise 45, with $S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 3 \\ 2 & 6 \end{bmatrix}$;

$$A = BS^{-1} = \frac{1}{5}\begin{bmatrix} 9 & 3 \\ -2 & 16 \end{bmatrix}.$$

49. Let A be the matrix of T and C the matrix of L. We want that $AP_0 = P_1$, $AP_1 = P_3$, and $AP_2 = P_2$. We can use the result of Exercise 45, with $S = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ and

$$B = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

Then $A = BS^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$.

Using an analogous approach, we find that $C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

51. Let E be an elementary $n \times n$ matrix (obtained from I_n by a certain elementary row operation), and let F be the elementary matrix obtained from I_n by the reversed row operation. Our work in Exercise 50 [parts (a) through (c)] shows that $EF = I_n$, so that E is indeed invertible, and $E^{-1} = F$ is an elementary matrix as well.

53. a. Let $S = E_1 E_2 \dots E_p$ in Exercise 52a.

By Exercise 51, the elementary matrices E_i are invertible: now use Fact 2.4.8 repeatedly to see that S is invertible.

b. $A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{matrix} \div 2 \\ \end{matrix}$, represented by $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \begin{matrix} \\ -4(I) \end{matrix}$, represented by $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

$\text{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

Therefore, $\text{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = E_1 E_2 A = SA$, where

$S = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -2 & 1 \end{bmatrix}$.

(There are other correct answers.)

55. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ represents a horizontal shear, $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ represents a vertical shear,

$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ represents a "scaling in $\vec{e}_1$ direction" (leaving the $\vec{e}_2$ component unchanged),

$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ represents a "scaling in $\vec{e}_2$ direction" (leaving the $\vec{e}_1$ component unchanged), and

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents the reflection about the line spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

57. Let A and B be two lower triangular $n \times n$ matrices. We need to show that the ijth entry of AB is 0 whenever $i < j$.

This entry is the dot product of the ith row of A and the jth column of B,

$$[a_{i1}\ a_{i2} \ldots a_{ii}\ 0 \ldots 0] \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{jj} \\ \vdots \\ b_{nj} \end{bmatrix}, \text{ which is indeed 0 if } i < j.$$

59. a. Write the system $L\vec{y} = \vec{b}$ in components:

$$\begin{vmatrix} y_1 & & & & & & & = & -3 \\ -3y_1 & + & y_2 & & & & & = & 14 \\ y_1 & + & 2y_2 & + & y_3 & & & = & 9 \\ -y_1 & + & 8y_2 & - & 5y_3 & + & y_4 & = & 33 \end{vmatrix}, \text{ so that } y_1 = -3,\ y_2 = 14 + 3y_1 = 5,$$

$y_3 = 9 - y_1 - 2y_2 = 2$, and $y_4 = 33 + y_1 - 8y_2 + 5y_3 = 0$:

$$\vec{y} = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 0 \end{bmatrix}.$$

b. Proceeding as in part (a) we find that $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$.

61. a. Write $L = \begin{bmatrix} L^{(m)} & 0 \\ L_3 & L_4 \end{bmatrix}$ and $U = \begin{bmatrix} U^{(m)} & U_2 \\ 0 & U_4 \end{bmatrix}$. Then $A = LU = \begin{bmatrix} L^{(m)}U^{(m)} & L^{(m)}U_2 \\ L_3U^{(m)} & L_3U_2 + L_4U_4 \end{bmatrix}$, so that $A^{(m)} = L^{(m)}U^{(m)}$, as claimed.

b. By Exercise 34, the matrices L and U are both invertible. By Exercise 2.3.35, the diagonal entries of L and U are all nonzero. For any m, the matrices $L^{(m)}$ and $U^{(m)}$ are triangular, with nonzero diagonal entries, so that they are invertible. By Fact 2.4.8, the matrix $A^{(m)} = L^{(m)}U^{(m)}$ is invertible as well.

c. Using the hint, we write $A = \begin{bmatrix} A^{(n-1)} & \vec{v} \\ \vec{w} & k \end{bmatrix} = \begin{bmatrix} L' & 0 \\ \vec{x} & t \end{bmatrix} \begin{bmatrix} U' & \vec{y} \\ 0 & s \end{bmatrix}$.

We are looking for a column vector $\vec{y}$, a row vector $\vec{x}$, and scalars t and s satisfying these equations. The following equations need to be satisfied: $\vec{v} = L'\vec{y}$, $\vec{w} = \vec{x}U'$, and $k = \vec{x}\vec{y} + ts$.

We find that $\vec{y} = (L')^{-1}\vec{v}$, $\vec{x} = \vec{w}(U')^{-1}$, and $ts = k - \vec{w}(U')^{-1}(L')^{-1}\vec{v}$.

We can choose, for example, $s = 1$ and $t = k - \vec{w}(U')^{-1}(L')^{-1}\vec{v}$, proving that A does indeed have an LU factorization.

Alternatively, one can show that if all principal submatrices are invertible then no row swaps are required in the Gauss-Jordan Algorithm. In this case, we can find an LU-factorization as outlined in Exercise 58.

63. We will prove that $A(C + D) = AC + AD$, repeatedly using Fact 1.3.9a: $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$.

    Write $B = [\vec{v}_1 \ \dots \ \vec{v}_m]$ and $C = [\vec{w}_1 \ \dots \ \vec{w}_m]$. Then

    $A(C + D) = A[\vec{v}_1 + \vec{w}_1 \cdots \vec{v}_m + \vec{w}_m] = [A\vec{v}_1 + A\vec{w}_1 \ \cdots \ A\vec{v}_m + A\vec{w}_m]$, and

    $AC + AD = A[\vec{v}_1 \ \cdots \ \vec{v}_m] + A[\vec{w}_1 \ \cdots \ \vec{w}_m] = [A\vec{v}_1 + A\vec{w}_1 \ \cdots A\vec{v}_m + A\vec{w}_m]$.

    The results agree.

65. Suppose A_{11} is a $p \times p$ matrix and A_{22} is a $q \times q$ matrix. For B to be the inverse of A we must have $AB = I_{p+q}$. Let us partition B the same way as A:

    $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, where B_{11} is $p \times p$ and B_{22} is $q \times q$.

    Then $AB = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$ means that

    $A_{11}B_{11} = I_p, \ A_{22}B_{22} = I_q, \ A_{11}B_{12} = 0, \ A_{22}B_{21} = 0.$

    This implies that A_{11} and A_{22} are invertible, and $B_{11} = A_{11}^{-1}, \ B_{22} = A_{22}^{-1}$.

    This in turn implies that $B_{12} = 0$ and $B_{21} = 0$.

    We summarize: A is invertible if (and only if) both A_{11} and A_{22} are invertible; in this case

    $A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}$.

67. Write A in terms of its rows: $A = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \cdots \\ \vec{w}_n \end{bmatrix}$ (suppose A is $n \times m$).

We can think of this as a partition into n

$1 \times m$ matrices. Now $AB = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \cdots \\ \vec{w}_n \end{bmatrix} B = \begin{bmatrix} \vec{w}_1 B \\ \vec{w}_2 B \\ \cdots \\ \vec{w}_n B \end{bmatrix}$ (a product of partitioned matrices).

We see that the ith row of AB is the product of the ith row of A and the matrix B.

69. Suppose A_{11} is a $p \times p$ matrix. Since A_{11} is invertible, $\text{rref}(A) = \begin{bmatrix} I_p & A_{12} & * \\ 0 & 0 & \text{rref}(A_{23}) \end{bmatrix}$, so that

$\text{rank}(A) = p + \text{rank}(A_{23}) = \text{rank}(A_{11}) + \text{rank}(A_{23})$.

71. Multiplying both sides with A^{-1} we find that $A = I_n$: The identity matrix is the only invertible matrix with this property.

73. We must find all S such that $SA = AS$, or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

So $\begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$, meaning that $b = 2b$ and $c = 2c$, so b and c must be zero.

We see that all diagonal matrices (those of the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$) commute with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

75. Again, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Thus, $\begin{bmatrix} 2b & -2a \\ 2d & -2c \end{bmatrix} = \begin{bmatrix} -2c & -2d \\ 2a & 2b \end{bmatrix}$, meaning that $c = -b$ and $d = a$.

We see that all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ commute with $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

77. Now we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Thus, $\begin{bmatrix} a+2b & 2a-b \\ c+2d & 2c-d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 2a-c & 2b-d \end{bmatrix}$. So $a + 2b = a + 2c$, or $c = b$, and $2a - b = b + 2d$, revealing $d = a - b$. (The other two equations are redundant.)

All matrices of the form $\begin{bmatrix} a & b \\ b & a-b \end{bmatrix}$ commute with $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

79. We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then, $\begin{bmatrix} a+2b & 3a+6b \\ c+2d & 3c+6d \end{bmatrix} = \begin{bmatrix} a+3c & b+3d \\ 2a+6c & 2b+6d \end{bmatrix}$. So $a+2b = a+3c$, or $c = \frac{2}{3}b$, and $3a+6b = b+3d$, revealing $d = a + \frac{5}{3}b$. The other two equations are redundant.

Thus all matrices of the form $\begin{bmatrix} a & b \\ \frac{2}{3}b & a+\frac{5}{3}b \end{bmatrix}$ commute with $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.

81. Now we want $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$,

or, $\begin{bmatrix} 2a & 3b & 2c \\ 2d & 3e & 2f \\ 2g & 3h & 2i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 2g & 2h & 2i \end{bmatrix}$. So, $3b = 2b$, $2d = 3d$, $3f = 2f$ and $3h = 2h$, meaning that b, d, f and h must all be zero.

Thus all matrices of the form $\begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & i \end{bmatrix}$ commute with $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

83. The ijth entry of AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$.

Then $\sum_{k=1}^{n} a_{ik}b_{kj} \le \sum_{k=1}^{n} sb_{kj} = s\left(\sum_{k=1}^{n} b_{kj}\right) \le sr$.

↑ ↑

since $a_{ik} \le s$ this is $\le r$, as it is the jth column sum of B.

85. a. The components of the jth column of the technology matrix A give the demands industry J_j makes on the other industries, per unit output of J_j. The fact that the jth column sum is less than 1 means that industry J_j *adds value* to the products it produces.

b. A productive economy can satisfy any consumer demand $\vec{b}$, since the equation

$(I_n - A)\vec{x} = \vec{b}$ can be solved for the output vector $\vec{x} : \vec{x} = (I_n - A)^{-1}\vec{b}$ (compare with Exercise 2.3.49).

c. The output $\vec{x}$ required to satisfy a consumer demand $\vec{b}$ is

$$\vec{x} = (I_n - A)^{-1}\vec{b} = (I_n + A + A^2 + \cdots + A^m + \cdots)\,\vec{b} = \vec{b} + A\vec{b} + A^2\vec{b} + \cdots + A^m\vec{b} + \cdots.$$

To interpret the terms in this series, keep in mind that whatever output $\vec{v}$ the industries produce generates an interindustry demand of $A\vec{v}$.

The industries first need to satisfy the consumer demand, $\vec{b}$. Producing the output $\vec{b}$ will generate an interindustry demand, $A\vec{b}$. Producing $A\vec{b}$ in turn generates an extra interindustry demand, $A(A\vec{b}) = A^2\vec{b}$, and so forth.

For a simple example, see Exercise 2.3.50; also read the discussion of "chains of interindustry demands" in the footnote to Exercise 2.3.49.

87. a. $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Matrix A^{-1} transforms a wife's clan into her husband's clan, and B^{-1} transforms a child's clan into the mother's clan.

b. B^2 transforms a women's clan into the clan of a child of her daughter.

c. AB transforms a woman's clan into the clan of her daughter-in-law (her son's wife), while BA transforms a man's clan into the clan of his children. The two transformations are different. (See Figure 2.24.)

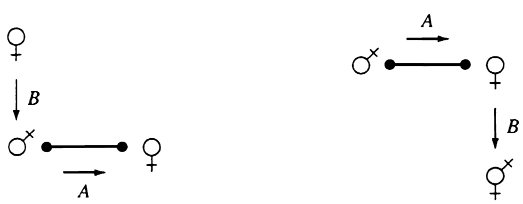

Figure 2.24: for Problem 2.4.87c .

d. The matrices for the four given diagrams (in the same order) are $BB^{-1} = I_3$,

$$BAB^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ B(BA)^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},\ BA(BA)^{-1} = I_3.$$

e. Yes; since $BAB^{-1} = A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, in the second case in part (d) the cousin belongs to Bueya's husband's clan.

89. $g(f(x)) = x$, for all x, so that $g \circ f$ is the identity, but $f(g(x)) = \begin{cases} x & \text{if } x \text{ is even} \\ x+1 & \text{if } x \text{ is odd} \end{cases}$.

True or False

1. T; The matrix is $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

3. T, by Fact 2.3.3.

5. F, by Fact 2.4.3.

7. F; Matrix AB will be 3×5, by Definition 2.4.1b.

9. T, by Fact 2.2.4.

11. F, by Fact 2.3.6. Note that the determinant is 0.

13. T; The shear matrix $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ works.

15. T; The equation $\det(A) = k^2 - 6k + 10 = 0$ has no real solution.

17. F; Note that $\det(A) = (k-2)^2 + 9$ is always positive, so that A is invertible for all values of k.

19. F; Consider $A = I_2$ (or any other invertible 2×2 matrix).

21. F; For any 2×2 matrix A, the two columns of $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ will be identical.

23. F; A reflection matrix is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2+b^2 = 1$. Here, $a^2+b^2 = 1+1 = 2$.

25. T; The product is $\det(A)I_2$.

27. T; Note that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a rotation through $\pi/2$. Thus $n = 4$ (or any multiple of 4) works.

29. F; If matrix A has two identical rows, then so does AB, for any matrix B. Thus AB cannot be I_n, so that A fails to be invertible.

31. F; Consider the matrix A that represents a rotation through the angle $2\pi/17$.

33. T; We have $(5A)^{-1} = \frac{1}{5}A^{-1}$.

35. T; Note that $A^2B = AAB = ABA = BAA = BA^2$.

37. F; Consider $A = I_2$ and $B = -I_2$.

39. F; Consider matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, for example.

41. T; If you reflect twice in a row (about the same line), you will get the original vector back: $A(A\vec{x}) = \vec{x}$, or, $A^2\vec{x} = \vec{x} = I_2\vec{x}$. Thus $A^2 = I_2$ and $A^{-1} = A$.

43. T; Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, for example.

45. T; We can rewrite the given equation as $A^2 + 3A = -4I_3$ and $-\frac{1}{4}(A + 3I_3)A = I_3$. By Fact 2.4.9, matrix A is invertible, with $A^{-1} = -\frac{1}{4}(A + 3I_3)$.

47. F; A and C can be two matrices which fail to commute, and B could be I_n, which commutes with anything.

49. F; Since there are only eight entries that are not 1, there will be at least two rows that contain only ones. Having two identical rows, the matrix will be non-invertible.

51. F; We will show that $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}S$ fails to be diagonal, for an arbitrary invertible matrix $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now, $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}S = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix}$. Since c and d cannot both be zero (as S must be invertible), at least one of the off-diagonal entries ($-c^2$ and d^2) is nonzero, proving the claim.

53. T; Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now we want $A^{-1} = -A$, or $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. This holds if $ad - bc = 1$ and $d = -a$. These equations have many solutions: for example, $a = d = 0, b = 1, c = -1$. More generally, we can choose an arbitrary a and an arbitrary nonzero b. Then, $d = -a$ and $c = -\frac{1+a^2}{b}$.

55. T; Recall from Definition 2.2.1 that a projection matrix has the form $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$, where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector. Thus, $a^2 + b^2 + c^2 + d^2 = u_1^4 + (u_1u_2)^2 + (u_1u_2)^2 + u_2^4 = u_1^4 + 2(u_1u_2)^2 + u_2^4 = (u_1^2 + u_2^2)^2 = 1^2 = 1$.