Math 253, Spring 2002, Exam #2

Exam # 2

1. **True or False?**

T F If the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ span $\mathbb{R}^3$, then the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ must form a basis of $\mathbb{R}^3$.

T F If the rank of a 7×10 matrix A is 4, then the kernel of A must be six-dimensional.

T F If V is the set of all 2×2 matrices A such that the vector $\begin{bmatrix}1\\2\end{bmatrix}$ is in the image of A, then V is a subspace of $\mathbb{R}^{2\times2}$.

T F For every subspace V of $\mathbb{R}^4$ there exists a 4×4 matrix A such that $V=\text{im}(A)$.

T F There exists a noninvertible 2×2 matrix A that is similar to $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$.

2. Are the functions below **isomorphisms?** You will earn 2 points for each correct answer, and 1 point if you don't answer. No explanation is needed. We are told that one (and only one) of these functions fails to be linear.

Yes N o $T(A)=SAS^{-1}$, where $S=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$.

Yes No $T\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5x+6y & 6x+7y\\7x+8y & 8x+9y\end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}^{2\times2}$

Yes No $T(f(x))=f(x)+3$ from P_2 to P_2.

Yes No $T(f(x))=f(0)+f(1)x+f(2)x^2$ from P_2 to P_2.

Yes No $T(f(x))=(x-1)f(x)$ from P to P.

3. Find a basis of the subspace V of P_3 consisting of all polynomials $f(x)$ with $f(1)=f(2)$. Find the dimension of V.

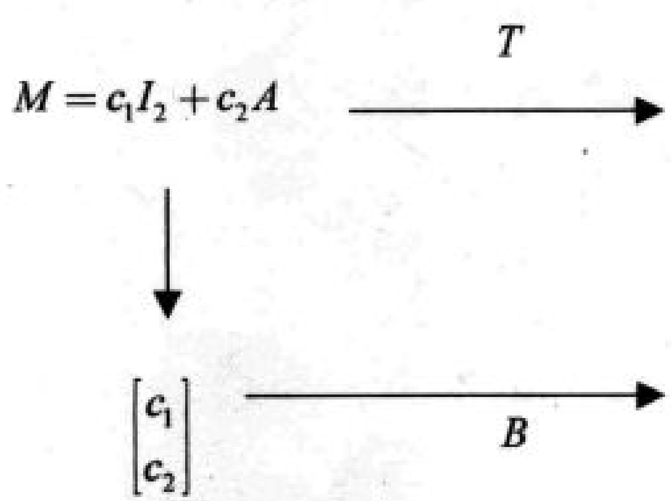
$f(1) = f(2)$. Find the dimension of V.

4. If $b \neq 0$, find the matrix B of the linear transformation $T(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ with respect to the basis $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$. Express the entries in the second column of B in terms of the determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the trace of A (the trace is the sum of the diagonal entries, $a + d$).

5. Let V be the span of the matrices I_2 and $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ in $\mathbb{R}^{2\times 2}$. Consider the linear transformation $T(M) = AM$ from V to V.

a. Compute A^2. Write your answer as a scalar multiple of matrix A.

b. Find the matrix B of T with respect to the basis $\mathfrak{B} = I_2, A$. Use the commutative diagram below.



c. Find a basis of the image of T

d. Find a basis of the kernel of T

Exam # 2, Solutions

1. **True or False?**

a. F As a counter example, consider $\vec{v}_1 = \vec{e}_1, \quad \vec{v}_2 = \vec{e}_2, \quad \vec{v}_3 = \vec{0}, \quad \vec{v}_4 = \vec{e}_3$

b. T $\dim(\ker A) = (\#\text{ columns}) - (\text{rank}A) = 10 - 4 = 6$

c. F The zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ isn't in V.

d. T Pick a basis $\vec{v}_1, \ldots, \vec{v}_m$ of V. Make $\vec{v}_1, \ldots, \vec{v}_m$ the first m columns of A, with the remaining columns (if any) all being $\vec{0}$ (or otherwise dependent on the $\vec{v}_i$).

e. F The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is invertible, and any matrix that is similar to an invertible matrix is invertible as well.

2. Are the functions below **isomorphisms**?

a. Yes The inverse is $A = S^{-1}BS$

b. No The dimensions of domain and codomain aren't equal.

c. No That's the nonlinear one; note that $T(0) = 3$

d. Yes The kernel is 0, since the only polynomial $f(x)$ in P_2 with $f(0) = f(1) = f(2) = 0$ is the zero polynomial.

e. No The image isn't all of P, but $\text{Im}(T) = \{g \text{ in } P : g(1) = 0\}$.

3. We are looking for the polynomials $f(x) = a + bx + cx^2 + dx^3$ such that $f(1) = f(2)$, or, $a + b + c + d = a + 2b + 4c + 8d$, or $b + 3c + 7d = 0$, or $b = -3c - 7d$. These polynomials are of the form $f(x) = a + (-3c - 7d)x + cx^2 + dx^3$ $= a \cdot 1 + c\left(x^2 - 3x\right) + d\left(x^3 - 7x\right)$, so that $a, x^2 - 3x, x^3 - 7x$ is a basis of V, and $\dim(V) = 3$.

4. With $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $S = \begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$, we have $B = S^{-1}AS = \frac{1}{b}\begin{bmatrix} -d & b \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$

$$= \begin{bmatrix} 0 & bc - ad \\ 1 & a + d \end{bmatrix} = \begin{bmatrix} 0 & -\det(A) \\ 1 & \text{trace}(A) \end{bmatrix}$$

$$= \begin{bmatrix} \cdots & \cdots \\ 1 & a+d \end{bmatrix} = \begin{bmatrix} \cdots & \cdots \\ 1 & \text{trace}(A) \end{bmatrix}$$

5. a.. $A^2 = \begin{bmatrix} 7 & 14 \\ 21 & 42 \end{bmatrix} = 7A$

b.

$$\begin{array}{ccc} M = c_1 I_2 + c_2 A & \xrightarrow{\quad T \quad} & T(M) = AM = c_1 A + c_2 A^2 = (c_1 + 7c_2)A \\ \downarrow & & \downarrow \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow[\quad B \quad]{} & \begin{bmatrix} 0 \\ c_1 + 7c_2 \end{bmatrix} \end{array}$$

Thus $B = \begin{bmatrix} 0 & 0 \\ 1 & 7 \end{bmatrix}$

c. A basis of the image of B is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and a basis of the image of T is A.

d. A basis of the kernel of B is $\begin{bmatrix} 7 \\ -1 \end{bmatrix}$, and a basis of the kernel of T is

$7I_2 - A = \begin{bmatrix} 6 & -2 \\ -3 & 1 \end{bmatrix}$.