Exam # 2, Math 253, Spring 2001

1. True or False?

a. The column vectors of a 4×5 matrix must be linearly dependent.
b. If a 3×3 matrix A is non-invertible, then the last column of A must be a linear combination of the first two columns of A.
c. If the rank of a 9×10 matrix A is 5, then the kernel of A is 4-dimensional.
d. If matrix A is similar to B, and A is invertible, then B must be invertible as well.
e. If A is an invertible 3×3 matrix and $\vec{x}$ is a non-zero vector in $\mathbb{R}^3$, then the vectors $\vec{x}, A\vec{x}$, and $A^2\vec{x}$ must form a basis of $\mathbb{R}^3$.

2. True or False?

a. The function $T(A)=A\begin{bmatrix}1&2\\3&4\end{bmatrix}$ is an isomorphism from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$.

b. The functions $f(x)$ in C^∞ such that $\int_{-1}^{1} f(x)dx=0$ form a subspace of C^∞.

c. The function $T(f)=f+f''$ is an isomorphism from C^∞ to C^∞.

d. The function $T(f)=\begin{bmatrix}f(1)&f(-1)\\f(2)&f(-2)\end{bmatrix}$ is an isomorphism from P_4 to $\mathbb{R}^{2\times2}$.

e. There exists a two-dimensional subspace V of $\mathbb{R}^{2\times2}$ such that all matrices in V are noninvertible.

3. Let V be the set of all polynomials $f(x)$ in P_3 such that $f'(0)=0$ and $f(2)=0$. We are told that V is a subspace of P_3. Find a basis of V and determine the dimension of V.

4. Consider the linear transformation $T\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_2\\x_1\end{bmatrix}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Is there a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix of T is $\begin{bmatrix}-1&2\\0&1\end{bmatrix}$? Find such a basis $\mathfrak{B}$, or show that none can exist.

5. Consider the function $T(M)=\begin{bmatrix}1&2\\0&3\end{bmatrix}M-M\begin{bmatrix}2&0\\3&k\end{bmatrix}$ from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$, where k is an arbitrary constant. We are told that T is a linear transformation.

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a. For the standard basis $\mathfrak{B}$ of $\mathbb{R}^{2\times 2}$, find the $\mathfrak{B}$-matrix B of T. Some of the entries of your matrix B will contain the constant k.

b. For which values of the constant k is T an isomorphism?

Math 253, Exam # 2, Spring 2001, Solutions

2. True or False?

T F The function $T(A)=A\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ is an isomorphism from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

True. Check linearity of T. Since the matrix $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ is invertible, we can write a formula for the inverse of the transformation $B=A\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, namely, $A=B\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}^{-1}$.

T F The functions $f(x)$ in C^∞ such that $\int_{-1}^{1} f(x)dx=0$ form a subspace of C^∞.

True. Check the three axioms for a subspace:

- The function $f(x)=0$ is the neutral element of C^∞, and $\int_{-1}^{1} 0\,dx=0$
- It's closed under addition: If $\int_{-1}^{1} f=0$ and $\int_{-1}^{1} g=0$, then $\int_{-1}^{1}(f+g)=\int_{-1}^{1} f+\int_{-1}^{1} g=0$
- It's closed under scalar multiplication: If $\int_{-1}^{1} f=0$, then $\int_{-1}^{1}(kf)=k\int_{-1}^{1} f=0$ for all k.

T F The function $T(f)=f+f''$ is an isomorphism from C^∞ to C^∞.

False. $\ker(T)=\{f / f''+f=0\}=\{f / f''=-f\}=\text{span}(\sin x,\cos x)\neq\{0\}$ (see Fact 4.2.3b)

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False. $\dim(P_4)=5$ but $\dim(\mathbb{R}^{2\times 2})=4$ (see Fact 4.2.3d)

T F There exists a subspace V of $\mathbb{R}^{2\times 2}$ such that all matrices in V are non-invertible.

True. For example, $V=\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a,b \text{ in } \mathbb{R}\right\}$

3) Let V be the set of all polynomials $f(x)$ in P_3 such that $f'(0)=0$ and $f(2)=0$. We are told that V is a subspace of P_3. Find a basis of V and determine the dimension of V.

Step 1: Write down a general element of the "ambient space", P_3, involving some arbitrary constants:

$$f(x)=a+bx+cx^2+dx^3$$

Step 2: Use the conditions that define subspace V to set up a linear system for the arbitrary constants in Step 1.

$$f'(x)=b+2cx+3dx^2 \text{ , so that } f'(0)=b=0$$
$$f(2)=a+2b+4c+8d=a+4c+8d=0$$

Thus the system (in rref) is

$$\begin{array}{ccccc} a & & +4c & +8d & =0 \\ & b & & & =0 \end{array}$$

Step 3 : Use Gaussian elimination to solve the system in Step 2 and write down the general element of V:

$$a=-4c-8d \quad \text{and} \quad b=0$$
$$\text{general element of } V: \quad f(x)=(-4c-8d)+cx^2+dx^3$$

Step 4: Write the general element in Step 3 as a linear combination, using the arbitrary constants as the coefficients. Check that the elements of V in this linear combination are linearly independent. Then they will form a basis of V:

$$f(x)=c\left(x^2-4\right)+d\left(x^3-8\right) \quad ,$$
$$\textbf{basis of } V: x^2-4, x^3-8,$$
$$\dim(V)=2$$

4) Consider the linear transformation $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Is there a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix of T is $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$? Find such a basis $\mathfrak{B}$, or show that none can exist.

Solution: Refer to the terminology introduced in Section 3.4, involving matrices A, B, and S.

We are told that $B = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$.

Note that $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We are looking for an invertible $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that $AS = SB$; the columns of S will then give us a basis of $\mathbb{R}^2$ as desired. Now consider the equation

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} z & t \\ x & y \end{bmatrix} = \begin{bmatrix} -x & 2x+y \\ -z & 2z+t \end{bmatrix}$$

We are left with two equations, $z = -x$ and $t = 2x + y$ (the other two are redundant), so that the general matrix S that solves the equation $AS = SB$ is of the form

$$S = \begin{bmatrix} x & y \\ -x & 2x+y \end{bmatrix}$$

We need to choose values for x and y that make S invertible; one possible choice is $x = 1, y = 0$. This gives $S = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ and the basis $\mathfrak{B}$ consisting of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

General solution: $\begin{bmatrix} x \\ -x \end{bmatrix}, \begin{bmatrix} y \\ 2x+y \end{bmatrix}$, as long as $x \neq 0$ and $y \neq -x$

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5) Consider the function $T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} M - M \begin{bmatrix} 2 & 0 \\ 3 & k \end{bmatrix}$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$, where k is an arbitrary constant. We are told that T is a linear transformation.

a) For the standard basis $\mathfrak{B}$ of $\mathbb{R}^{2\times 2}$, find the $\mathfrak{B}$-matrix B of T. Some of the entries of your matrix B will contain the constant k.

Solution:

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & k \end{bmatrix}$$

$$= \begin{bmatrix} a+2c & b+2d \\ 3c & 3d \end{bmatrix} - \begin{bmatrix} 2a+3b & kb \\ 2c+3d & kd \end{bmatrix} = \begin{bmatrix} -a-3b+2c & (1-k)b+2d \\ c-3d & (3-k)d \end{bmatrix}$$

Now write input and output in coordinates with respect to the standard basis.

$$\begin{array}{ccc}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} & \longrightarrow & \begin{bmatrix} -a-3b+2c & (1-k)b+2d \\ c-3d & (3-k)d \end{bmatrix} \\
\big\downarrow & & \big\downarrow \\
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} & \longrightarrow & \begin{bmatrix} -a-3b+2c \\ (1-k)b+2d \\ c-3d \\ (3-k)d \end{bmatrix}
\end{array}$$

Thus

$$B = \begin{bmatrix} -1 & -3 & 2 & 0 \\ 0 & (1-k) & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & (3-k) \end{bmatrix}, \text{ an upper triangular matrix}$$

b) For which values of the constant k is T an isomorphism?

Solution: T is an isomorphism iff matrix B is invertible. Use either the determinant or Gaussian elimination to see that a triangular matrix is invertible iff all diagonal entries are nonzero.

Thus T is an isomorphism iff the constant k is neither 1 nor 3.