Math 253, Fall 2001, Final Exam, Solutions

1. a. **T** by definition of the image: The vector $\vec{b}$ is in the image of A iff there is a vector $\vec{x}$ such that $\vec{b} = A\vec{x}$.

b. **T**. There is a nontrivial relation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Then $A(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = c_1A\vec{v}_1 + c_2A\vec{v}_2 + c_3A\vec{v}_3 = \vec{0}$ is a nontrivial relation as well, proving the claim.

c. **F** The determinants aren't the same (5 versus –5).

d. **T** $\det(AB) = (\det A)(\det B) \neq 0$, so that $\det(A) \neq 0$ as well.

e. **F** Counter example $\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$.

2. a. **T** Consider five eigenvectors $\vec{v}_1, .., \vec{v}_5$ with distinct, nonzero eigenvalues $\lambda_1, ..., \lambda_5$. Vectors $\vec{v}_1, .., \vec{v}_5$ are independent (Fact 7.3.5), and they are in the image of A (since $\vec{v}_k = A\left(\frac{1}{\lambda_k}\vec{v}_k\right)$, so that $\operatorname{rank}(A) = \dim(\operatorname{im} A) \geq 5$ as claimed.

b. **F** Counter example $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

c. **T** There are seven distinct eigenvalues, namely, $0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}$.

d. **T** The eigenspace E_1 must be all of $\mathbb{R}^n$, so that $A\vec{v} = \vec{v}$ for all $\vec{v}$, meaning that A is the identity matrix I_n.

e. **F**. As a counter example, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Note that $A^2 = 0$, so that any nonzero vector is an eigenvector of A^2, with eigenvalue 0.

3. $B = S^{-1}AS = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$, where $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$, $S = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, and $S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$.

4. Expand along the second row:

$$\det(A) = -a_{23}\det(A_{23}) = -6\det\begin{vmatrix} 1 & -1 & 0 \\ 3 & 0 & 4 \\ \end{vmatrix} = (-6)(-20+18) = 12.$$

$$\det(A) = -a_{23}\det(A_{23}) = -6\det\begin{vmatrix} 3 & 0 & 4 \\ 0 & 5 & 6 \end{vmatrix} = (-6)(-20+18) = 12.$$

5. Denote the columns by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_5$, and look for relations among the columns. By inspection, we find that $\vec{v}_2 = 2\vec{v}_1$ (or $2\vec{v}_1 - \vec{v}_2 = \vec{0}$) and $\vec{v}_5 = 3\vec{v}_1 + 4\vec{v}_3 - 3\vec{v}_4$ (or $3\vec{v}_1 + 4\vec{v}_3 - 3\vec{v}_4 - \vec{v}_5 = \vec{0}$). The remaining vectors $\vec{v}_1, \vec{v}_3, \vec{v}_4$ are independent.

a. The "remaining vectors" $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ form a basis of the image.

b. The relations we found above give us a basis of the kernel: $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 0 \\ 4 \\ -3 \\ -1 \end{bmatrix}$.

6. a. Consider the signs of the terms $-2q(t)$ and $p(t)$ to see that the q's are the wolfs.

b. $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} = \vec{x}(t) = A^t\vec{x}(0)$, where $A = \begin{bmatrix} 6 & -2 \\ 1 & 3 \end{bmatrix}$ and $\vec{x}(0) = \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = \begin{bmatrix} 600 \\ 500 \end{bmatrix}$. The eigenvalues of A are 5 and 4, with the corresponding eigenspaces $E_5 = \ker\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} = \text{span}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $E_4 = \ker\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} = \text{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenbasis for A. Now express the initial state vector as a linear combination of the eigenvectors: $\vec{x}(0) = \begin{bmatrix} 600 \\ 500 \end{bmatrix} = 100\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 400\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We conclude that

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} = \vec{x}(t) = A^t\vec{x}(0) = 100A^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 400A^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 100(5^t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 400(4^t)\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$ so that

$p(t) = 200(5^t) + 400(4^t)$ and $q(t) = 100(5^t) + 400(4^t)$.

7. a. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow L(A) = 2A + 3A^T = \begin{bmatrix} 5a & 2b+3c \\ 3b+2c & 5d \end{bmatrix}$

$\downarrow \qquad \qquad \downarrow$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \xrightarrow{\quad B \quad} \begin{bmatrix} 5a \\ 2b+3c \\ 3b+2c \\ 5d \end{bmatrix}$$

Thus $$B = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

b. Use the determinant or rref to see that B is invertible ($\det(B) = 5^2 \det \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} = -125$).

Thus L is an isomorphism.

c. Since L is an isomorphism, the image of L is all of $\mathbb{R}^{2\times 2}$. Thus I_2 is in $\text{im}(L)$.

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8. a. We are looking for the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a+2b \\ c+2d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

or, $a = -2b, c = -2d$. The general element of V is $\begin{bmatrix} -2b & b \\ -2d & d \end{bmatrix}$

$= b\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$, so that $\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$ is a basis of V, and $\dim(V) = 2$.

b. The space V we studied in part a is a subspace of W (if $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in the kernel of A, then $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A, with eigenvalue 0). Since W is neither V nor all of $\mathbb{R}^{2\times 2}$ (think about it!), we have $V \not\subset W \not\subset \mathbb{R}^{2\times 2}$ and

(think about it!), we have $V \not\subset W \not\subset \mathbb{R}^{2\times 2}$ and
$2 = \dim(V) < \dim(W) < \dim\left(\mathbb{R}^{2\times 2}\right) = 4$, so that $\dim(W) = 3$.

9. a. We are looking for a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with positive entries such that

$$\det(A - \lambda I_2) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc) = (\lambda-4)(\lambda-6)$$

$= \lambda^2 - 10\lambda + 24$. It is required that $\operatorname{trace}(A) = a + d = 10$ and $\det(A)$ $= ad - bc = 24$ (compare coefficients!). One possible solution is $a = d = 5$, $b = c = 1$. Thus $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ does the job.

b. There is no such matrix. A matrix B with four positive entries will have two distinct eigenvalues (use the quadratic formula!), so that there will be an eigenbasis for B.

10. Recall that a *nonzero* polynomial in P_n has at most n zeros. **Thus the zero polynomial is the only polynomial with infinitely many zeros.**

a. If $f(x)$ is in the kernel of T, then $T(f(x)) = 0$, so that $f(0) = f(1) = \ldots = f(n) = \ldots = 0$ for all positive integers n. Thus $f(x) = 0$ is the zero polynomial, by the preliminary remark. It follows that $\ker(T) = \{0\}$.

b. We claim that the infinite sequence $(1,0,0,\ldots.,0,\ldots)$ fails to be in the image of T (this sequence starts with a 1, with all other entries being 0). Indeed, there is no polynomial $f(x)$ with $T(f(x)) = (f(0), f(1), f(2), \ldots, f(n), \ldots) = (1,0,0,\ldots,0,\ldots)$, or $f(0) = 1, f(1) = f(2) = \ldots. = f(n) = \ldots = 0$, by our preliminary remark.
Thus T fails to be an isomorphism.