UC Berkeley Summer Undergraduate Research Program 2015
July 9 Lecture

We will introduce the basic structure and representation theory of the symplectic group $\mathrm{Sp}(V)$.

Basics Fix a nondegenerate, alternating bilinear form $\omega : V \times V \to \mathbb{C}$, where V is a finite dimensional $\mathbb{C}$-vector space. This means that

1. $\omega(u, v) = -\omega(v, u)$
2. $\omega(u, V) = 0 = \omega(V, u) \implies u = 0 \in V$,
3. $\omega(au + bv, w) = a\omega(u, w) + b\omega(v, w)$.

Let $e_1 \neq 0$. Then, by (2) there is some $f_1 \notin \mathrm{span}(e_1)$ such that $\omega(f_1, e_1) \neq 0$; scaling we may assume that $\omega(f_1, e_1) = 1$. Let $U_1 = \mathrm{span}(e_1, f_1) \subset V$. The subspace

$$U_1^{\perp} = \{v \in V \mid \omega(u, v) = 0, \ \text{ for all } u \in U_1\} \subset V$$

satisfies $U_1 \cap U_1^{\perp} = \{0\}$ (why?). Nondegeneracy ensures an isomorphism

$$V \to V^* \ ; \ v \mapsto \omega(-, v),$$

so that we can identify the annihilator $U_1^{\vee} \subset V^*$ with some $\dim V - 2$ subspace $V_1 \subset V$ via this isomorphism. Moreover, $U_1 \cap V_1 = \{0\}$, and $\omega_{|V_1}$ is nondegenerate: if $\omega(v, V_1) = 0$ then $\omega(v, V) = 0 \implies v = 0$. Hence, by induction we can assume that V_1 admits a basis $(e_2, \dots, e_n, f_n, \dots, f_2)$ such that $\omega(f_i, e_j) = \delta_{ij}$. Then, with respect to the basis $(e_1, \dots, e_n, f_n, \dots, f_1)$, the matrix of ω is

$$J \overset{def}{=} [\omega] = \begin{bmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{bmatrix}$$

Remark: $J^{-1} = -J = J^t$

The symplectic group (with respect to ω) $\mathrm{Sp}(V, \omega)$ is the group of all operators on V preserving ω

$$\mathrm{Sp}(V, \omega) = \{g \in \mathrm{GL}(V) \mid \omega(gu, gv) = \omega(u, v), \ \text{for all } u, v \in V\}.$$

We have just seen above that any nondegenerate alternating form ω admits a basis such that the matrix of ω is J: this means that all such pairs (V, ω) (such a pair is defined as a **symplectic vector space**; a corresponding basis is called a **symplectic basis**) are **equivalent**, in the sense that, for any two such pairs $(V, \omega), (V', \omega')$, there is an isomorphism

$$T : V \to V', \quad \text{such that } \omega'(Tu, Tv) = \omega(u, v).$$

In particular, we may as well choose $V = \mathbb{C}^{2n}$, and ω to be the nondegenerate, alternating form

$$\omega_0 : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C} \ ; \ (u, v) \mapsto u^t J v.$$

Hence, we can restrict ourselves to the group

$$\mathrm{Sp}(\mathbb{C}^{2n}, \omega_0) = \{g \in \mathrm{GL}_{2n} \mid g^t J g = J\}.$$

We will denote this group Sp_{2n}: it is a subvariety of GL_{2n} defined by the equations $g^t J g = J$. Hence, Sp_{2n} is an affine algebraic variety such that its group operations are given by morphisms of algebraic varieties. Moreover, we see that

$$\det(g^t J g) = \det J \implies \det(g)^2 = 1 \implies \det(g) = \pm 1.$$

Exercise:

1. If $g \in \mathrm{Sp}_{2n}$ then $g^t \in \mathrm{Sp}_{2n}$.

2. Let $g \in \mathrm{Sp}_{2n}$, $v \in V$, such that $gv = \lambda v$. Show that there exists $w \in V$ such that $gw = \lambda^{-1} w$. Deduce that $\mathrm{Sp}_{2n} \subset \mathrm{SL}_{2n}$.

Tori, Weyl Group It can be checked that the intersection $T = \mathrm{Sp}_{2n} \cap T_{2n}$, with the standard maximal torus in GL_{2n}, is

$$T = \left\{ \begin{bmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{bmatrix} \mid t_1, \dots, t_n \in \mathbb{C}^\times \right\} \cong (\mathbb{C}^\times)^n$$

Hence, T is a complex torus. Moreover, T is maximal.

You can think of T as being those operators on $\mathbb{C}^{2n}$ that preserve ω_0 and for which the **symplectic basis** is a common eigenbasis.

Let's 'guess' what we expect the **Weyl group** $W = N_{\mathrm{Sp}_{2n}}(T)/T$ to be:

- For GL_n we saw that the normaliser $N_G(T)$ consists of all those operators that preserved the set of lines defined by the common eigenbasis for $T \subset \mathrm{GL}_n$. When we divided out by the action of T we forgot about the scaling that we could have within each line, so that $N_G(T)/T$ was the group acting on the set of n eigenlines.

- You might guess that we simply permute all elements of the symplectic basis but this is not correct: if we swapped e_1 and e_2 then we would no longer preserve the form ω_0 as $1 = \omega_0(f_1, e_1) \neq \omega_0(f_1, e_2) = 0$. A moment's thoughtand we see that permuting e_i and e_j means that we must also permute f_i and f_j. Hence, there should be a copy of S_n sitting inside W: it consists of those symmetries that preserve the pairs of lines $(\mathbb{C}e_i, \mathbb{C}f_i)$.

- We are allowed further symmetry: within each pair of lines $(\mathbb{C}e_i, \mathbb{C}f_i)$, we can swap the lines. However, when we swap the lines we must map $\mathbb{C}e_i$ to $-\mathbb{C}f_i$, in order that the symmetry preserve ω_0.

- Hence, we should think of the Weyl group of Sp_{2n} as being the symmetry we have in any ordering of the 'symplectic planes' we choose. We expect there to be subgroups of W isomorphic to S_n - corresponding to swapping the pairs of lines $(\mathbb{C}e_i, \mathbb{C}f_i)$ - and $(\mathbb{Z}/2\mathbb{Z})^n$ - swapping the lines within each pair (with the 'minus twist').

Fact: the Weyl group of Sp_{2n} is

$$W(C_n) \stackrel{def}{=} W \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n,$$

where $\sigma \in S_n \subset W$ acts on $(a_1, \dots, a_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ by permuting entries.

For example, the representatives of the symmetric group appearing in W are of the form

$$\begin{bmatrix} \sigma & 0 \\ 0 & \underline{\sigma} \end{bmatrix}$$

where $\underline{\sigma}$ is the permutation matrix obtained by **reflecting σ in the antidiagonal**. For example, if $\sigma = (12) \in S_3$ then we have

$$\begin{bmatrix} & 1 & & & & \\ 1 & & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & & 1 \\ & & & & 1 & \end{bmatrix}.$$

The element $(a_1, \dots, a_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ has representative a $2n \times 2n$ matrix $[c_{ij}]$ with

$$c_{ii} = c_{2n+1-i,2n+1-i} = 1, \quad \text{whenever } a_i = 0 \in \mathbb{Z}/2\mathbb{Z},$$

$$c_{2n+1-i,i} = 1 = -c_{i,2n+1-i}, \quad \text{whenever } a_i = 1 \in \mathbb{Z}/2\mathbb{Z}.$$

For example, in Weyl group $W(C_2)$ we have the following representatives for $(12) \in S_2$, $(1,0) \in (\mathbb{Z}/2\mathbb{Z})^2, (0,1) \in (\mathbb{Z}/2\mathbb{Z})^2$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise: $W(C_2) \cong D_8$. However, $W(C_n) \ncong D_{4n}$ in general (why?).

Roots etc The character lattice of T is generated by the projections onto the i^{th} diagonal entry. Denote these projections χ_i, so that $X^*(T) = \sum \mathbb{Z}\chi_i$.

We are going to determine the root system - for this we need to know the Lie algebra of Sp_{2n}. It is a fact that

$$\mathfrak{sp}_{2n} \stackrel{def}{=} \{X \in M_{2n} \mid X^t J + JX = 0\} = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid {}^tC = C, {}^tB = B, -D = {}^tA \right\}$$

where ${}^tM$ denotes **the transpose across the antidiagonal**.

For example, we have

$$\mathfrak{sp}_4 = \left\{ \begin{bmatrix} a & b & e & f \\ c & d & g & e \\ h & i & -d & -b \\ j & h & -c & -a \end{bmatrix} \right\}$$

$$\mathfrak{sp}_6 = \left\{ \begin{bmatrix} a & b & c & x_{11} & x_{12} & z \\ d & e & f & x_{21} & y & x_{12} \\ g & h & i & x & x_{21} & x_{11} \\ y_{11} & y_{12} & w & -i & -f & -c \\ y_{21} & v & y_{12} & -h & -e & -b \\ u & y_{21} & y_{11} & -g & -d & -a \end{bmatrix} \right\}$$

Then, Sp_{2n} acts linearly on $\mathfrak{sp}_{2n}$ by conjugation, and T admits a common eigenbasis: namely, they are the same basis vectors as we see for the above examples. For example, for $\mathfrak{sp}_4$ we have eigenbasis

$$\{E_{11}-E_{44}, E_{22}-E_{33}, E_{12}-E_{34}, E_{21}-E_{43}, E_{13}+E_{24}, E_{14}, E_{23}, E_{32}, E_{41}, E_{31}+E_{42}\}$$

and the weights are

$$0, \pm\chi_1-\chi_2, \pm\chi_1+\chi_2, \pm 2\chi_1, \pm 2\chi_2$$

If we let $\alpha_1=\chi_1-\chi_2$, $\alpha_2=2\chi_2$, then we can write the nonzero elements above as

$$\pm\alpha_1,\ \pm(\alpha_1+\alpha_2),\ \pm(2\alpha_1+\alpha_2),\ \pm\alpha_2;$$

these are the **roots of** $\mathfrak{sp}_4$.

In the $\mathfrak{sp}_6$ case we have weights

$$0,\ \pm\chi_i-\chi_j,\ \pm\chi_k+\chi_l, \quad i<j,\ k\leq l.$$

If we let

$$\alpha_1=\chi_1-\chi_2,\ \alpha_2=\chi_2-\chi_3,\ \alpha_3=2\chi_3,$$

then we can write the above nonzero weights

$$\pm\alpha_i,\ \pm(\alpha_1+\alpha_2),\ \pm(\alpha_2+\alpha_3),\ \pm(2\alpha_2+\alpha_3)\ \pm(\alpha_1+\alpha_2+\alpha_3),\ \pm(\alpha_1+2\alpha_2+\alpha_3),$$

$$\pm(\alpha_1+2\alpha_2+\alpha_3),\ \pm(2\alpha_1+2\alpha_2+\alpha_3);$$

these are the **roots of** $\mathfrak{sp}_6$.

In general, the roots of Sp_{2n} (with respect to our choice of T) are

$$R \stackrel{def}{=} \{\chi_i-\chi_j \mid i\neq j\}\cup\{\chi_i+\chi_j \mid i\leq j\}\subset X^*(T)$$

and a set of simple roots are

$$\alpha_1=\chi_1-\chi_2,\ \alpha_2=\chi_2-\chi_3,\ \dots\ ,\ \alpha_n=2\chi_n.$$

This allows us to define the **positive roots**

$$R_+ \stackrel{def}{=} \{\alpha\in\mathbb{R} \mid \alpha=\sum n_i\alpha_i,\ n_i\in\mathbb{Z}_{\geq 0}\},$$

and the **negative roots** $-R_+$.

For $\alpha\in R$, a **root subgroup** is a subgroup

$$U_\alpha=I_{2n}+aE_\alpha, \quad a\in\mathbb{C}$$

where E_α is an eigenvector with T-eigenvalue α.

Proposition/Definition: The standard Borel $B\subset\mathrm{Sp}_{2n}$ is the subgroup generated by T and U_α, $\alpha\in R_+$. It is a Borel subgroup. It is the intersection $\mathrm{Sp}_{2n}\cap B_{2n}$ of Sp_{2n} with the upper triangular matrices in GL_{2n}.

We define the **fundamental weights**

$$\omega_1=\chi_1,\ \omega_2=\chi_1+\chi_2,\ \dots,\ \omega_{n-1}=\chi_1+\dots+\chi_{n-1},\ \omega_n=\chi_1+\dots+\chi_n.$$

The corresponding **Dynkin diagram** is

$$\bullet \; - - - \; \bullet \; - - - \; \bullet \; \cdots \; \bullet \; \Longleftarrow \; \bullet$$

This signifies that the angle between α_i and α_{i+1} is $2\pi/3$, for $i = 1, \dots, n-1$; the angle between α_{n-1} and α_n is $3\pi/4$; and all other angles between simple roots is $\pi/2$. All of these angles are measured in the Euclidean space $\sum \mathbb{R}\chi_i$, with respect to the standard inner (= dot) product.

The set of roots R is a root system: the reflections defined by each simple root α_i are

$$s_i : \; v \mapsto v - 2\frac{(v, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i.$$

The group generated by the s_i in $\mathrm{GL}(X^*(T))$ is isomorphic to the Weyl group.

Remark: Observe, that $\mathrm{rank}\mathbb{Z}R = n = \dim T$, in contrast to the GL_n case: this is because Sp_{2n} is **(semi)simple**.

Representation Theory The main Theorem is

Theorem:

1. Let W be a finite dimensional irreducible representation of Sp_{2n}. Then, there is a unique line $L \subset W$ that is B-invariant, and on which T acts by $\lambda \in X^*(T)$. Moreover, if $\mu \in \Lambda(W)$ is a weight then $\lambda \geq \mu$; we call λ **the highest weight in** W. The weight $\lambda = \sum_i a_i \chi_i$ satisfies $a_1 \geq \dots \geq a_n \geq 0$.

2. To any weight $\lambda = \sum_i a_i \chi_i$, $a_1 \geq \dots \geq a_n \geq 0$, there is an irreducible representation of Sp_{2n} with λ as its highest weight.

We will focus on the example Sp_4.