

UC Berkeley Summer Undergraduate Research Program 2015

July 7 Lecture

Today we will see how to use representation theory to produce embeddings of the flag variety (and partial flag varieties) inside some projective space \mathbb{P}^M .

Fix $G = GL_n$ and let $T \subset B \subset G$ be the diagonal matrices sitting inside the upper triangular matrices.

Recall that the irreducible (finite dimensional) representations of G are indexed by partitions $\lambda: \lambda_1 \geq \dots \geq \lambda_n \geq 0$. In general, it's not so easy to determine V_λ ! However, we have seen that the representations $\wedge^k \mathbb{C}^n$ are irreducible with highest weight

$$\omega_i \stackrel{\text{def}}{=} \chi_1 + \dots + \chi_k :$$

we will now write $V_{\omega_k} = \wedge^k \mathbb{C}^n$. These weights are called the **fundamental weights**. Observe that if we define the 1-parameter subgroup

$$\alpha_i^\vee : t \mapsto \text{diag}(1, \dots, t, t^{-1}, 1, \dots, 1), \quad i = 1, \dots, n-1,$$

then $\omega_k \circ \alpha_i^\vee(t) = \delta_{ik}$.

Fix $1 \leq k \leq n$. We know that the line $\mathbb{C}e_1 \wedge \dots \wedge e_k \subset V_{\omega_k}$ is B -invariant: this is the same as saying that the point $[e_1 \wedge \dots \wedge e_k] \in \mathbb{P}(V_{\omega_k}) \cong \mathbb{P}^{\binom{n}{k}-1}$ is fixed by the action of B on $\mathbb{P}(V_{\omega_k})$ (why is this well-defined action?). However, via the induced action of G on $\mathbb{P}(V_{\omega_k})$, it can be seen that

$$\text{Stab}_G([e_1 \wedge \dots \wedge e_k]) = \left\{ \begin{bmatrix} * & * \\ 0 & *_{n-k} \end{bmatrix} \right\},$$

a **parabolic subgroup** of G . Denote this parabolic subgroup P_k . It is a fact that P_k is a **maximal parabolic subgroup**.

Hence, we get an identification

$$G/P_k \cong G \cdot [e_1 \wedge \dots \wedge e_k] \subset \mathbb{P}(V_{\omega_k}).$$

Recall the weight basis $\{e_{i_1} \wedge \dots \wedge e_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq n\}$ of V_{ω_k} . For a subset $J \subset \{1, \dots, n\}$ of cardinality k , denote the element of the dual basis $x_J \in V_{\omega_k}^*$. Hence, we can consider

$$\mathbb{P}(V_{\omega_k}) \longleftrightarrow \mathbb{P}^{\binom{n}{k}-1}; [v] \longleftrightarrow [x_J(v)].$$

For example, when $n = 4$ we have $V_{\omega_2} = \wedge^2 \mathbb{C}^4$ and we get an isomorphism

$$\mathbb{P}(V_{\omega_2}) \longleftrightarrow \mathbb{P}^5; [v] \longleftrightarrow [x_{12}(v) : x_{13}(v) : x_{14}(v) : x_{23}(v) : x_{24}(v) : x_{34}(v)]$$

In particular, we identify

$$\begin{aligned} [e_1 \wedge e_2] &\leftrightarrow [1 : 0 : 0 : 0 : 0 : 0], \\ [e_1 \wedge e_3] &\leftrightarrow [0 : 1 : 0 : 0 : 0 : 0] \quad \text{etc.} \end{aligned}$$

We have already seen how to identify G/P_k with $\text{Gr}(k, n)$ (although in a slightly different way): we can think of a k -dimensional subspace $U \subset \mathbb{C}^n$ as the **column-span** of the first k columns of an invertible matrix $A = [a_1 \dots a_n]$, so $U = \text{col}(a_1, \dots, a_k)$.

Given a matrix A we will write $\text{col}_k(A)$ for the column span of the first k columns.

If we define an action of G on $\text{Gr}(k, n)$ by

$$g \cdot U \stackrel{\text{def}}{=} \text{col}_k(gA), \quad \text{where } U = \text{col}_k(A),$$

the stabiliser of $U = \text{span}(e_1, \dots, e_k)$ is P_k . **Note that this action is well-defined.** Hence, as the action of G is transitive we obtain $\text{Gr}(k, n) \cong G/P_k$. We can realise this equivalence of G -sets through the map

$$G \rightarrow \text{Gr}(k, n); \quad g \mapsto g \cdot \text{span}(e_1, \dots, e_k).$$

Namely, we can factor

$$\begin{array}{ccc} G & \rightarrow & \text{Gr}(k, n) \\ \downarrow & \nearrow & \\ G/P_k & & \end{array}$$

The inverse of the dashed arrow is the map

$$\text{Gr}(k, n) \rightarrow G/P_k; \quad U = \text{col}_k(A) \mapsto AP_k,$$

This is well-defined.

Remark: we have had to be careful here so that we get an identification of the **left cosets** gP_k **with** k -**dimensional subspaces of** \mathbb{C}^n .

Given a subset $J = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ of cardinality k , and an $n \times n$ matrix A , we can form the $k \times k$ minor $\Delta_J(A)$ with columns $1, \dots, k$ and rows i_1, \dots, i_k . As we saw in the first week for $n = 3, k = 2$, we can **coordinatise** $\text{Gr}(k, n)$ using these **Plucker coordinates**

$$\text{Gr}(k, n) \rightarrow \mathbb{P}^{\binom{n}{k}-1}; \quad U = \text{col}_k(A) \mapsto [\Delta_J(A)].$$

In particular, the **Plucker embedding** above is injective. (*We can even give a description of the image of this map as a projective variety*)

Let's see how these coordinates arrive in the representation-theoretic framework: denote the natural pairing between a vector space W and its dual W^* by

$$\langle \cdot, \cdot \rangle : W^* \times W \rightarrow \mathbb{C}; \quad (\alpha, v) \mapsto \langle \alpha, v \rangle.$$

Consider the following functions on G :

$$\xi_J : g \mapsto \langle x_J, g(e_1 \wedge \dots \wedge e_k) \rangle.$$

If we consider the representation of G defined by V_{ω_k} , then $\xi_J(g)$ is the $(J, 1)$ entry of the matrix of the action of g on V_{ω_k} , with respect to the weight basis.

Since V_{ω_k} is a rational representation, the $\xi_J \in A_G$ are regular functions on G , and are P_k -invariant in the sense that $\xi_J(gp) = \xi_J(g)$, for any $p \in P_k$. Thus, they descend to the quotient G/P_k , so that we obtain functions $\xi_J : G/P_k \rightarrow \mathbb{C}$.

If the columns of g are g_1, \dots, g_n then we have

$$\xi_J(g) = \langle x_J, g_1 \wedge \dots \wedge g_k \rangle = \Delta_J(g).$$

One way to check this without performing a calculation is to use the following characterisation of the determinant: there exists a unique function $f : \mathbb{C}^k \times \dots \times \mathbb{C}^k \rightarrow \mathbb{C}$ (k copies of \mathbb{C}^k , though of as row vectors) such that

1. f is alternating, $f(u_1, \dots, u_i, \dots, u_j, \dots, u_k) = -f(u_1, \dots, u_j, \dots, u_i, \dots, u_k)$,
2. f is multilinear, ie, it is linear in each of its arguments,
3. $f(e_1, \dots, e_k) = 1$.

In summary: **the Plucker embedding of the Grassmannian has an intrinsic characterisation (without reference to minors). The Grassmannian is the image in $\mathbb{P}(V_{\omega_k})$ of the morphism**

$$G/P_k \rightarrow \mathbb{P}(V_{\omega_k}); gP_k \mapsto [gu],$$

where $u \in V_{\omega_k}$ is any highest weight vector (so that $tu = \omega_k(t)u$).

We observe that the homogenous coordinate ring of G/P_k under this embedding is isomorphic to the **Plucker algebra** $\mathbb{C}[\Delta_J]$.

Now, let $\lambda = \sum_i \lambda_i \chi_i$ be a weight such that $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ and $u \in V_\lambda$ a highest weight vector (so that $tu = \lambda(t)u$, for $t \in T$). If we set $a_i = \lambda_i - \lambda_{i+1} \geq 0$, $a_n = \lambda_n$, then we can write

$$\lambda = a_1\omega_1 + a_2\omega_2 + \dots + a_n\omega_n.$$

The action of G on V_λ is linear so that it descends to an action on $\mathbb{P}(V_\lambda)$

$$g \cdot [v] \stackrel{\text{def}}{=} [gv].$$

Recall that $[v]$ is B -invariant, so that $B \subset \text{Stab}_G([v])$ and $P_\lambda \stackrel{\text{def}}{=} \text{Stab}_G([v])$ is a parabolic subgroup of G . Hence, G/P_λ is a projective variety and we have determined an embedding

$$G/P_\lambda \rightarrow \mathbb{P}(V_\lambda); gP_\lambda \mapsto [gv].$$

Fact: if $\lambda = a_1\omega_1 + \dots + a_n\omega_n$ and $I = \{i \mid a_i \neq 0\} = \{d_1 < \dots < d_k\}$ then $\text{Stab}_G([v])$ is a parabolic subgroup consisting of block upper triangular matrices of size $d_1, d_2 - d_1, \dots, d_k - d_{k-1}$. Moreover, G/P_λ is isomorphic to the multistep Grassmannian $\text{Gr}(d_1, d_2, \dots, d_k)$. In particular, whenever $I = \{1, \dots, n\}$ we obtain an embedding of the full flag variety G/B in some projective space.