

UC Berkeley Summer Undergraduate Research Program 2015

July 2 Lecture

We introduced some of the basic structure theory for the general linear group $G = \text{GL}(V)$, where V is a finite dimensional \mathbb{C} -vector space. We introduced several classes of subgroups of G :

$$T, U \subset B \subset P \subset G$$

and introduced some representations for G , the exterior powers $\bigwedge^k V$. Note that these representations have the property that there exists a simultaneous eigenbasis for T ; in fact, this is always the case. We will discuss the internal structure of a representation of G today by investigating what the representation looks like as a representation of T , U and B (and B_- (the opposite standard Borel)).

Highest weight theory and roots: Fix a maximal torus T in GL_n - we may as well assume that this is the standard torus, since all maximal tori are conjugate. The **group of characters of T** (also called the **weight lattice**) is the set of group homomorphisms

$$X^*(T) \stackrel{\text{def}}{=} \text{Hom}_{\text{alg. gp}}(T, \mathbb{C}^\times),$$

that are also morphisms of algebraic varieties. We have already seen that $X^*(T) \cong \mathbb{Z}^n$: in fact, we observe that the projections $x_{jj} : T \rightarrow \mathbb{C}^*$ provide a basis for the free abelian group $X^*(T)$. This is the same as saying that $A_T \cong \mathbb{C}[x_{11}^\pm, \dots, x_{nn}^\pm]$. We will denote these (special) characters $\chi_j \stackrel{\text{def}}{=} x_{jj}$.

Fact: for any representation (W, ρ) , every element $\rho(t) \in \text{GL}(W)$, $t \in T$, is diagonalisable.

Hence, since $\rho(T) \subset \text{GL}(W)$ is a commutative subgroup consisting of diagonalisable elements, we can find a simultaneous eigenbasis $\{w_1, \dots, w_m\}$ of W . This means that

$$t \cdot w_i = \alpha_i(t) w_i,$$

where $\alpha_i : T \rightarrow \mathbb{C}^\times$ takes $t \in T$ to the eigenvalue $\alpha_i(t)$ of the linear operator $\rho(t)$ associated with the eigenvector w_i . Furthermore, since

$$t \cdot (t' \cdot w_i) = (tt') \cdot w_i,$$

we have $\alpha_i \in X^*(T)$. The set $\Lambda(W) \stackrel{\text{def}}{=} \{\alpha_i\} \subset X^*(T)$ is called **the set of weights of W** ; elements of the eigenbasis are called **weight vectors**.

1. Let \mathbb{C}^n be the defining representation of GL_n , (e_1, \dots, e_n) the standard basis. Then, in any of the wedge products $W_{(k)} = \bigwedge^k \mathbb{C}^n$, the basis of 'standard' wedges $\{e_{i_1} \wedge \dots \wedge e_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq n\}$ is a simultaneous eigenbasis; the set of weights is

$$\Lambda(W_{(k)}) = \left\{ \sum_{j=1}^k \chi_{i_j} \mid \{i_1, \dots, i_k\} \subset [n] \right\}$$

since

$$\begin{aligned} t \cdot e_{i_1} \wedge \dots \wedge e_{i_k} &= t_{i_1} \cdots t_{i_k} e_{i_1} \wedge \dots \wedge e_{i_k} \\ &= \chi_{i_1}(t) \cdots \chi_{i_k}(t) e_{i_1} \wedge \dots \wedge e_{i_k} = (\chi_{i_1} + \dots + \chi_{i_k})(t) e_{i_1} \wedge \dots \wedge e_{i_k} \end{aligned}$$

for any $t = \text{diag}(t_1, \dots, t_n)$. Note that we are using additive notation for the (abelian) group of characters.

2. **(Important Example)** Consider the **adjoint representation**

$$\text{Ad} : G \rightarrow \text{GL}(M_n) ; g \mapsto [X \mapsto gXg^{-1}]$$

You can check that this is a well-defined representation: it's linear, and a morphism of algebraic varieties.

The matrices E_{ij} , with 0s everywhere except for a 1 in the ij -entry, are a simultaneous eigenbasis for T . For example, we have

$$t \cdot E_{ij} = tE_{ij}t^{-1} = t_it_j^{-1}E_{ij} = (\chi_i - \chi_j)(t)E_{ij}, \quad t = \text{diag}(t_1, \dots, t_n) \in T$$

This representation is fundamental: it can be shown that we can identify M_n with the **Lie algebra** \mathfrak{gl}_n of G , the tangent space of G at the identity. This representation can be thought of as the derivative of the action of G on itself by conjugation. It's nice that it turns out be conjugation again!

Due to their central role in the representation theory of G , we elements in $\Lambda(M_n)$ the **roots of G** , and denote $R = \Lambda(M_n)$.

The weights $\alpha_i \stackrel{\text{def}}{=} \chi_i - \chi_{i+1} \in X^*(T)$, $i = 1, \dots, n-1$, which we'll call the **simple roots of G (relative to B)**, play an important role: observe that

$$\chi_i - \chi_j = \alpha_i + \dots + \alpha_{j-1}, \quad \text{whenever } i < j,$$

and

$$\chi_i - \chi_j = -(\alpha_i + \dots + \alpha_{j-1}), \quad \text{whenever } j < i.$$

In particular, **every root of G can be written as an integral sum $\sum_{\alpha \in R} n_\alpha \alpha$, with every n_α either nonnegative, or nonpositive.**

The lattice generated by R inside $X^*(T)$, denoted Σ , is called the **root lattice of G** : observe that the root lattice is not equal to the character lattice. In fact, Σ sits inside the lattice $\{\sum_i a_i \chi_i \mid \sum_i a_i = 0\}$.

Fact: consider the \mathbb{R} -span $\Sigma_{\mathbb{R}}$ of Σ , sitting inside the \mathbb{R} -span of X^* . The roots $R \subset \Sigma_{\mathbb{R}}$ are a (reduced) root system:

- (a) $\text{span}_{\mathbb{R}}(R) = \Sigma_{\mathbb{R}}$,
- (b) if $\alpha \in R$, and $c\alpha \in \mathbb{R}$, then $c = \pm 1$,
- (c) R is preserved under the linear operators

$$s_\alpha : v \mapsto v - 2 \frac{(\alpha, v)}{(\alpha, \alpha)} \alpha, \quad \alpha \in R$$

- (d) the real numbers $2 \frac{(\alpha, \beta)}{(\alpha, \alpha)}$ are integers.

For $G = \text{GL}_n$, the numbers $2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} \in \{0, \pm 1\}$. Moreover, s_α is the reflection through the hyperplane orthogonal to α in $\Sigma_{\mathbb{R}}$.

Given $\alpha = \chi_i - \chi_j \in R$, $i \neq j$, define the **root subgroup** $U_\alpha \subset G$ to be the unipotent group

$$U_\alpha = I_n + \mathbb{C}E_{ij}.$$

The root subgroups are subgroups isomorphic to \mathbb{C} , normalised by T so that

$$t(I_n + cE_{ij})t^{-1} = I_n + \alpha(t)cE_{ij}.$$

The representation theory of G can be described as follows:

1. every irreducible representation V of G admits a unique line $L = \text{span}(v) \subset V$, with v a weight vector of weight λ , with $\lambda = \sum_i \lambda_i \chi_i$, and where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ is a nondecreasing sequence of integers; we call λ the highest weight of V . Moreover, to any such sequence of integers $\lambda = (\lambda_1, \dots, \lambda_n)$ there is an irreducible representation with highest weight λ . We denote this irreducible representation by V_λ
2. the weights appearing in an irreducible representation V_λ are of the form

$$\mu = \lambda - \sum_i n_i \alpha_i, \quad n_i \in \mathbb{Z}_{\geq 0}.$$