

## UC Berkeley Summer Undergraduate Research Program 2015 June 23-24

This is an exposition of the paper: Kogan-Miller 'Toric Degenerations of Schubert varieties and Gelfand-Tsetlin polytopes', Adv. Math 193 (2005) p1-17.

By considering the Plucker algebra  $\mathcal{P}$  and exhibiting a SAGBI basis for  $\mathcal{P}$  (consisting of Plucker coordinates), we obtained a toric degeneration of the Flag variety (algebraically, at least).

In Kogan-Miller they construct the same toric degeneration (ie, the special fibre is the toric variety coming from the Gelfand-Tsetlin polytope). Moreover, their construction allows to see what happens to certain subvarieties of the flag variety under this degeneration - these are the Schubert varieties (=closures of Schubert cells), and are important in understanding the topology of flag varieties.

One of the projects for this summer is to try and perform a similar construction as in Kogan-Miller for a toric degeneration of a Grassmannian.

Again, we will focus on an example in order to highlight the main points.

Let  $M = M_3(\mathbb{C})$ , the set of  $3 \times 3$  matrices,  $G = GL_3$ ,  $B \subset G$  the set of lower triangular matrices in  $G$ .

There is an action of  $G^3 = G \times G \times G$  on  $M$  by column-wise multiplication - this means that the usual action of  $G$  on  $M$  is obtained via restriction to *the diagonal subgroup of  $G^3$* . Namely, if  $g = (g_1, g_2, g_3) \in G^3$  and  $m = [v_1 \ v_2 \ v_3] \in M$  then we define

$$g \cdot m \stackrel{\text{def}}{=} [g_1 v_1 \ g_2 v_2 \ g_3 v_3].$$

Denote by  $\underline{B} \subset G^3$  the image of  $B$  under the diagonal embedding of  $G$  in  $G^3$  - so elements of  $\underline{B}$  are of the form  $(b, b, b)$ , for  $b \in B$ .

Fix the following (generically arbitrary)  $3 \times 3$  matrix

$$W = [w_{ij}] = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is a homomorphism from  $\mathbb{C}^* = \{z \in \mathbb{C} \mid z \neq 0\}$ , the multiplicative group of nonzero complex numbers, into  $G^3$

$$T : \mathbb{C}^* \rightarrow G^3 ; z \mapsto (Z_1, Z_2, Z_3),$$

where  $Z_j = \text{diag}(z^{w_{1j}}, z^{w_{2j}}, z^{w_{3j}})$ . Denote  $T_W \stackrel{\text{def}}{=} \text{im } T \subset G^3$ , and if  $z \in \mathbb{C}^*$  we will write  $\tilde{z} \stackrel{\text{def}}{=} T(z) \in T_W$

**Exercise: is  $T_W$  a subgroup of  $\underline{B}$ ? Write down some elements in  $T_W$ .**

Define a **family of subgroups**  $B^* \subset B^3 \times \mathbb{C}^*$  as follows:

$$B^* \stackrel{\text{def}}{=} \{(\tilde{z}^{-1} b \tilde{z}, z) \mid b \in \underline{B}, z \in \mathbb{C}^*\}.$$

The fibre over  $z \in \mathbb{C}^*$  is denoted  $B(z) \cong \underline{B} \cong B$ . (**Show this last group isomorphism.**)

**Exercise:** check that if  $b = (b_1, b_2, b_3) \in B^3$  then  $\tilde{z}^{-1}b\tilde{z}$  is the element of  $B^3$  obtained from multiplying entries in  $b$  by the sequence

$$\begin{bmatrix} 1 & & \\ z^2 & 1 & \\ z^3 & z & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ z & 1 & \\ z & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix}. \quad (*)$$

For example, if

$$b = \begin{bmatrix} 1 & & \\ -2 & 2 & \\ 1 & 1 & 1 \end{bmatrix},$$

then

$$\tilde{z}^{-1}(b, b, b)\tilde{z} = \left( \begin{bmatrix} 1 & & \\ -2z^2 & 2 & \\ z^3 & z & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ -2z & 2 & \\ z & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ -2 & 2 & \\ 1 & 1 & 1 \end{bmatrix} \right) \in B(z)$$

Note that each power of  $z$  appearing in the sequence (\*) is positive; hence, we can also extend our family 'over 0', in the sense that we can set

$$B(0) = \{((b_1, b_2, b_3), 0)\},$$

where  $b_j$  obtained from  $b_3$  by setting to 0 all entries in columns  $1, \dots, 3-j$  strictly below main diagonal. That is, we are considering  $\lim_{z \rightarrow 0} B(z) = B(0)$ .

For example, we have

$$\left( \begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ -2 & 2 & \\ 1 & 1 & 1 \end{bmatrix} \right) \in B(0)$$

Denote  $p : B^* \rightarrow \mathbb{C}$  the resulting projection.

**Lemma:** there is an isomorphism of 'families of groups'  $B^* \cong B \times \mathbb{C}$  in the following sense - there is a function  $f : B \times \mathbb{C} \rightarrow B^*$  such that  $p \circ f = \pi$ , where  $\pi$  is the canonical projection to  $\mathbb{C}$ , and  $f$  restricts to group isomorphisms on fibres over  $\mathbb{C}$ .

*Proof:* Note that  $B \cong \underline{B}$ , sending  $b \mapsto \underline{b} = (b, b, b)$ . For  $0 \neq z \in \mathbb{C}$  we define

$$f(b, z) = (\tilde{z}^{-1}\underline{b}\tilde{z}, z) \in B^*.$$

If  $z = 0$  then we denote

$$f(b, 0) = ((b_1, b_2, b), 0) \in B(0),$$

where  $b_i$  is obtained from  $b$  by setting to 0 all entries below main diagonal in columns  $1, \dots, 3-i$  (as above). **(Check this is a group homomorphism on fibres: this means that you fix  $z \in \mathbb{C}$  and check that  $f(bb', z) = f(b, z)f(b', z)$ , where the group operation on the right comes from the group operation on  $G^3$ .)**

In fact, we have

$$f \left( \begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, z \right) = \left( \begin{bmatrix} a_{11} & & \\ a_{21}z^2 & a_{22} & \\ a_{31}z^3 & a_{32}z & a_{33} \end{bmatrix}, \begin{bmatrix} a_{11} & & \\ a_{21}z & a_{22} & \\ a_{31}z & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right)$$

We consider  $B$  acting on  $M \times \mathbb{C}$  by identifying  $B \times \mathbb{C}$  with  $B^*$  through the above isomorphism and consider  $B^*$  acting 'fibrewise' on  $M \times \mathbb{C}$ : ie we define an action

$$b * (m, z) \stackrel{\text{def}}{=} ((\bar{z}^{-1} \underline{b} \bar{z}) \cdot m, z)$$

For example, if

$$b = \begin{bmatrix} 1 & & \\ -2 & 2 & \\ 1 & 1 & 1 \end{bmatrix},$$

and  $X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$  then

$$\begin{aligned} b * (X, z) &= \left( \left( \begin{bmatrix} 1 & & \\ -2z^2 & 2 & \\ z^3 & z & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2z & 2 & \\ z & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 2 & \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right), z \right) \\ &= \left( \begin{bmatrix} 1 & & 0 \\ -2z^2 + 2 & -2z & 0 \\ z^3 + z & z + 1 & 1 \end{bmatrix}, z \right) \end{aligned}$$

**Remark:** When  $z = 0$ , we have that the action of  $B$  on  $M \times \{0\}$  commutes with the action of the subgroup

$$S = \left\{ \left( \begin{bmatrix} a & & \\ & b & \\ & & 1 \end{bmatrix}, \begin{bmatrix} c & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right) \in G^3 \right\} \subset G^3$$

on  $M$ . We will denote by  $\bar{t} \in S$  the sequence

$$\bar{z} \stackrel{\text{def}}{=} \left( \begin{bmatrix} z^3 & & \\ & z & \\ & & 1 \end{bmatrix}, \begin{bmatrix} z & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right) \in S.$$

We are now going to consider a 'family of Plucker coordinates': for  $J = \{j_1, \dots, j_k\} \subset \{1, 2, 3\}$ ,  $k = 1, 2, 3$ , we denote

$$w_J = \sum_{i=1}^2 w_{i, 4-j_i},$$

where  $W = [w_{ij}]$  is as above. For example,  $w_{12} = 0 + 0$ ,  $w_{13} = 0 + 1$ ,  $w_{23} = 1 + 1$ .

We are going to define 'Plucker coordinates on the family'  $M \times \mathbb{C}$  - these will be polynomials  $q_J(\underline{x}, z) \in \mathbb{C}[x_{11}, x_{12}, \dots, x_{32}, x_{33}, z]$  (think of the variable  $z$  as the degenerating parameter) such that, if we fix  $z = z_0$ , the polynomial  $q_J(\underline{x}, z_0)$  is  $U$ -invariant by restricting the  $*$ -action of  $B$  to

$$U = \left\{ \begin{bmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{bmatrix} \right\} \subset B.$$

This means that  $q_J(u * (m, z)) = q_J(m, z)$ , for any  $u \in U$ .

For  $J \subset \{1, 2, 3\}$ , we define

$$q_J(\underline{x}, z) = z^{-w_J} \Delta_J(\bar{z} \cdot X).$$

For example,

$$q_{12}(\underline{x}, z) = z^0 \Delta_{12}(\bar{z} \cdot X) = \Delta_{12} \left( \begin{bmatrix} z^3 x_{11} & zx_{12} & x_{13} \\ zx_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \right) = z^3 x_{11} x_{22} - z^2 x_{12} x_{21}.$$

**Important Theorem:** The polynomials  $q_J$  generate the  $\mathbb{C}[z]$ -algebra of  $U$ -invariant functions inside  $\mathbb{C}[\underline{x}, z]$ .