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Today we will discuss: complete flags, Plucker coordinates, Gelfand-Tsetlin polytopes.

In order to illustrate what's to come we will focus on an example and highlight some interesting features; all of what follows can discussed in general terms. See Miller-Sturmfels 'Combinatorial Commutative Algebra', Ch. 14 for more details. (*Note: this reference can be hard to read in places. I have extracted the essential results.*)

A **complete flag** in $\mathbb{C}^3$ (or simply a flag) is a sequence of subspace $\{0\} = V_0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3$, such that $\dim V_i = i$. We denote the set of all flags in $\mathbb{C}^3$ by Fl_3, and will often write $V_\bullet$ when considering a single flag.

So, really, a flag is just a pair (line, plane), with the line lying in the plane. We want to consider a 'nice' algebraic description of Fl_3; namely, we want some 'coordinates' that can distinguish different flags.

Suppose that $V_\bullet \in \mathrm{Fl}_3$ is a flag. Then, V_1 is determined by specfiying, up to nonzero scalar multiplication, a nonzero vector of V_1. Determining V_2 requires some more consideration. Take any basis (v_1, v_2) of V_2. Then, basic linear algebra states that

$$V_2 = \mathrm{row}\left(\begin{bmatrix} v_1^t \\ v_2^t \end{bmatrix}\right) = \mathrm{row}\, U,$$

where U is the reduced echelon form of the matrix $A = \begin{bmatrix} v_1^t \\ v_2^t \end{bmatrix}$; that is, row operations preserve the row space.

Hence, two 2-d subspaces of $\mathbb{C}^3$, V_2 and V_2', are equal precisely if the matrices A and A' we obtain after choosing a basis for each are row-equivalent.

This sounds like we've solved our problem (which we have, kind of). However, row-reduction is a pain in general (especially if we want to generalise our approach to flags in $\mathbb{C}^n$). As is the case in mathematics, let's make things harder to make them easier in the long run.

Notice that the possible reduced echelon forms of A are

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In each case we see that at least one of the 2×2-minors is nonzero; in fact, this is true in general - **let A be a 2×3 matrix. Then,** $\dim \mathrm{row}(A) = 2$ **if and only if at least one of the 2×2 minors is nonzero**.

Denote the minors of a 2×3 *generic* matrix $\Delta_{12}, \Delta_{13}, \Delta_{23}$; these are polynomial functions on the space of 2×3 matrices in the variables x_{ij}. Hence, for any $A = [a_{ij}] \in M_{2\times 3}(\mathbb{C})$,

$$\Delta_{12}(A) = a_{12}a_{22} - a_{21}a_{12}, \quad \text{etc.}$$

In fact, the minors completely determine the row span of a full-rank 2×3 matrix in the following sense: $\mathrm{row}(A) = \mathrm{row}(A')$ **if and only if there exists $c \neq 0$ such that** $(\Delta_{12}(A), \Delta_{13}(A), \Delta_{23}(A)) = c(\Delta_{12}(A'), \Delta_{13}(A'), \Delta_{23}(A'))$**.**

Let's see how this works: suppose that there's nonzero c as in the statement. As A and A' are full rank we must have that one of the minors is nonzero. Suppose that $\Delta_{12}(A) \neq 0$ (so that $\Delta_{12}(A') \neq 0$: we want to show that $\text{row}(A) = \text{row}(A')$. Choose bases (v_1, v_2) (resp. (v_1', v_2')) of $\text{row}(A)$ (resp. $\text{row}(A')$). We must show that the following systems of equations are consistent

$$A^t x = v_1', \quad A^t x = v_2'.$$

Suppose that

$$A^t = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix},$$

and B is the (2×2) inverse of $\begin{bmatrix} a & d \\ b & e \end{bmatrix}$.

We want to row-reduce $[A^t \ : \ (A')^t]$: we find

$$\begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} [A^t \ : \ (A')^t] = \begin{bmatrix} 1 & 0 & : & a' & d' \\ 0 & 1 & : & b' & e' \\ c & f & : & c' & f' \end{bmatrix}$$

The system we were solving is consistent if $(c, f) = (0, 0)$ implies $(c', f') = (0, 0)$. Suppose that $(c, f) = (0, 0)$. Then, $\Delta_{13}(A) = \Delta_{23}(A) = 0$, so that $\Delta_{13}(A') = \Delta_{23}(A') = 0$. Hence, $a'f' = d'c'$ and $b'f' = c'e'$. As $\Delta_{12}(A') \neq 0$ this implies that $(a', d') \neq (0, 0)$ (why?). If $a' \neq 0$ then $f' = d'c'/a'$ and

$$b'c'd'/a' = c'e' \implies 0 = \frac{c'}{a'}(b'd' - a'e') \implies 0 = \Delta_{12}(A'),$$

which is a contradiction. Similar arguments (assuming $\Delta_{13}(A) \neq 0, \Delta_{23}(A) \neq 0$) give the result.

Hence, up to nonzero scalar multiplication, a 2-d subspace in $\mathbb{C}^3$ is determined by (the nonzero vector) $(\Delta_{12}, \Delta_{13}, \Delta_{23})$. Hence, we've shown that there is an injective function

$$\{\text{2-d subspaces in } \mathbb{C}^3\} \to \mathbb{C}^3/\sim; \ V_2 \mapsto [\Delta_{ij}(A)],$$

where $u \sim v$ if there exists nonzero λ such that $u = \lambda v$. This is a particular example of a more general result that we will see later (**the Plucker embedding of a Grassmannian**).

Denote $\Delta_1 = x_{11}, \Delta_2 = x_{12}, \Delta_3 = x_{13}$.

Consider a flag $V_\bullet$. As above, we can use a 2×3 matrix A to write down $V_\bullet$ in more concrete terms: the first row of A spans V_1 and $\text{row}(A) = V_2$. Moreover, up to nonzero scalar multiplication, we find an injective function

$$\text{Fl}_3 \to \mathbb{C}^3/\sim \times \mathbb{C}^3/\sim; \ V_\bullet \mapsto ([\Delta_i(A)], [\Delta_{ij}(A)])$$

and we observe that the 'coordinates' Δ_i, Δ_{jk} are related to each other (they both come from a generic 2×3 matrix). There is exactly one relation (*Not so easy to see this!*) among the Δ's:

$$\Delta_1\Delta_{23} - \Delta_2\Delta_{13} + \Delta_3\Delta_{12} = 0.$$

Edit 6/22: there were some great observations today about 'orthogonal' lines etc. so I though I would include them here.

Remark: The above relation looks very much like a 'dot product'. In fact, there is a sense in which this is true: the subspace V_2 has a 1-d annihilator $V_2^\perp \subset (\mathbb{C}^3)^*$. Recall that the annihilator of a subspace U is

$$U^\perp = \{\alpha \in (\mathbb{C}^3)^* \mid \alpha(u) = 0, \text{ for every } u \in U\}.$$

The Plucker coordinates can be consider as defining a function from $\mathrm{Gr}(2,3) \to (\mathbb{C}^3)^*/\sim$ (I will provide a problem set outlining this tomorrow) taking a 2-d subspace to its annihilator (a line in $(\mathbb{C}^3)^*$). Then, $V_1 = \mathrm{span}(v_1)$ is a subspace of V_2 precisely when $\alpha(v_1) = 0$, where $\mathrm{span}(\alpha) = V_2^\perp$. REmember that elements of the dual of $\mathbb{C}^3$ should be considered as row vectors, so that $\alpha(v_1) = 0$ can be realised as a dot product.

Some Algebra

We can express the above information algebraically as follows: there is an algebra homomorphism

$$\Phi : \mathbb{C}[p_i, p_{jk}] \to \mathbb{C}[X] \; ; \begin{array}{ccc} p_i & \mapsto & \Delta_i \\ p_{jk} & \mapsto & \Delta_{jk} \end{array}$$

and we have $J \stackrel{def}{=} \ker \Phi = (p_1 p_{23} - p_2 p_{13} + p_3 p_{12})$.

We define $\mathcal{P} = \mathrm{im}\Phi$, the **Plucker (or flag) algebra**; it provides our first example of **toric degeneration** (whatever this means!). The Δ's appearing above are called **Plucker coordinates**.

Define a total order on $\mathbb{C}[p_i, p_{jk}]$ as follows: firs declare that

$$p_{12} \prec p_{13} \prec p_{23} \prec p_1 \prec p_2 \prec p_3,$$

and extend to the **grevlex order**: thus $p^a > p^b$ if and only if $|a| > |b|$ or $|a| = |b|$ and the rightmost nonzero entry of $a - b$ is negative. Here p^a is a monomial in the variables $p_1, p_2, \ldots, p_{23}$. Examples.

In particular, the **initial term of** $\underline{p_1 p_{23}} - p_2 p_{13} + p_3 p_{12}$ with respect to $\prec$ is $p_1 p_{23}$.

It is a fact from theory of Groebner bases that the monomials appearing outside of $\mathrm{in}_\prec(J)$ define a basis of $\mathbb{C}[p_i, p_{jk}]/J$.

Hence, a $\mathbb{C}$-basis of the Plucker algebra is given by the monomials

$$\{\Delta_1^*, \Delta_2^*, \Delta_3^*, \Delta_{12}^*, \Delta_{13}^*, \Delta_{23}^*, (\Delta_2\Delta_{13})^*, (\Delta_3\Delta_{12})^*\}$$

Now, we turn our attention to $\mathcal{P}$ proper. Order the variables x_{ij} by

$$x_{11} > x_{12} > x_{13} > x_{21} > x_{22} > x_{23},$$

and extend to an order on monomials in $\mathbb{C}[x]$ via lexicographic ordering. Notice that the initial (=highest) terms of a Plucker coordinate Δ is its **diagonal term**.

The Plucker coordinates form a **SAGBI basis** (=Subalgebra Analog of Groebner Basis for Ideals) for the Plucker algebra: they generate $\mathcal{P}$ and, moreover, their initial terms generate the **initial algebra of** $\mathcal{P}$ **(wrt** $<$**)**. Existence of SAGBI bases lead to nice normal forms for elements in $\mathcal{P}$.

For any monomial $\Phi(p^a) \in \mathcal{P}$, its initial term (in the x's) gives rise to a semistandard tableaux. Conversely, all monomials appearing in $\mathrm{in}_<(\mathcal{P})$ come from semistandard monomials. Examples.

The content of what we have seen above can be summarised as follows: **we can degenerate the Plucker algebra to the semigroup algebra generated by the semigroup $\mathcal{A}$ consisting of semistandard monomials.**

Gelfand-Tsetlin semigroups

Remark: Observe that the preceeding discussion depended on a choice: we chose an ordering on monomials so that the diagonal terms of the Plucker coordinates were initial. We showed that the Plucker coordinates then determined a SAGBI basis of $\mathcal{P}$ and this then allowed us to deduce that the semistandard monomials formed a basis for the initial algebra of $\mathcal{P}$ with respect to this (diagonal) ordering. In fact, if we chose an antidiagonal ordering, the same result holds (with appropriate modifications).

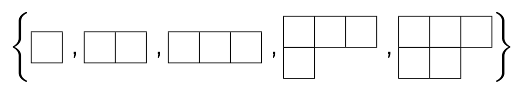
For a diagonal monomial order on $\mathbb{C}[x]$ we can describe the semigroup $\mathcal{A}$ in a nice combintorial manner. We represent the diagonal terms of the Plucker coordinates Δ via their positions in the matrix; for example

$$\Delta_1 \leftrightarrow \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}, \quad \Delta_{13} \leftrightarrow \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}.$$

Observe that we can put the shapes of these matrices (ie, the shapes of where the 1's are) into bijection with the set

$$\mathcal{H} = \{\text{partitions having (at most two) distinct parts of size at most 3}\}$$

Hence, we see that

$\mathcal{H} =$ 

A **Gelfand-Tsetlin pattern** is a sequence of (nonnegative) real numbers (a, b, c, u, v, w) satisfying some conditions (that I'll write on the board):

$$\begin{matrix} a & & b & & (c) \\ & u & & v & \\ & & w & & \end{matrix}$$

Consider the collection of integer GT patterns; they form a semigroup $\mathcal{GT}$ under componentwise addition.

Here's the culmination of the above considerations: **the semigroup $\mathcal{A}$ is isomorphic (as a semigroup), to the semigroup of Gelfand-Tsetlin patterns $\mathcal{GT}$.**

Remark: this is a little different to what appears in Miller-Sturmfels. This is because our definition of GT-pattern is 'not quite' correct; however, for our current purposes this doesn't matter.

So, what have we just seen? Here's a summary:

- We can use Plucker coordinates to give a straightforward(?) description of complete flags; namely, two flags are the same if and only if their Plucker coordinate values are the same (for any choice of matrix to represent them).

- Thus, it seems natural to study the algebra of Plucker coordinates (this is what algebraic geometry is about), so we introduced the Plucker algebra $\mathcal{P}$.

- Plucker coordinates generate $\mathcal{P}$ and have a nice property - they form a SAGBI basis. This allows us to 'degenerate' $\mathcal{P}$ to a simpler semigroup algebra (= algebra generated by monomials in x's).

- The resulting semigroup algebra (ie the algebra generated by monomials with exponents appearing in $\mathcal{A}$) is isomorphic to the algebra generated by the semigroup of Gelfand-Tsetlin patterns; this is an algebra generated by monomials.

Here's the punchline: the algebraic degeneration we have obtained above gives rise to a geometric degeneration of the flag variety to a **toric variety**