WHY ALL RINGS SHOULD HAVE A 1

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1. Introduction

Should the definition of ring require the existence of a multiplicative identity 1?

Emmy Noether, when giving the modern axiomatic definition of a commutative ring, in 1921, did not include such an axiom.¹ For several decades, algebra books followed suit.² But starting around 1960, many books by notable researchers³ began using the term "ring" to mean "ring with 1". Sometimes a change of heart occurred in a single person, or between editions of a single book, always towards requiring a 1.⁴ Reasons were not given; perhaps it was just becoming increasingly clear that the 1 was needed for many theorems to hold.⁵

But is either convention more natural? The purpose of this article is to answer yes, and to give a reason: existence of a 1 is a part of what associativity should be.

2. Total associativity

The whole point of associativity is that it lets us assign an unambiguous value to the product of any finite sequence of two or more terms. By why settle for "two or more"? Cognoscenti do not require *sets* to have two or more elements. So why restrict attention to sequences with two or more terms? Most natural would be to require *every* finite sequence to have a product, even if the sequence is of length 1 or 0:

Definition. A product on a set A is a rule that assigns to each finite sequence of elements of A an element of A, such that the product of a 1-term sequence is the term. A product is **totally** associative if each finite product of finite products equals the product of the concatenated sequence (for example, (abc)d(ef) should equal the 6-term product abcdef).

The ring axioms should be designed so that they give rise to a totally associative product. Now the key point is the following theorem, the more involved direction of which (the "if" direction) is [Bou70, I.§1.2, Théorème 1, and §2.1].

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¹See [Noe21, p. 29]. Noether was preceded by David Hilbert in 1897 and Adolf Fraenkel in 1914, who used the word *ring* in more restrictive senses. Hilbert used *Zahlring*, *Ring*, and *Integrätsbereich* to mean what we would call a finitely generated subring of an algebraic closure of \mathbb{Q} [Hil97, §31]; he implicitly included a 1. Fraenkel explicitly required a 1, but the concept he was axiomatizing was quite different from the modern concept of ring: for example, his axiom R_{8} required that every non-zerodivisor have a multiplicative inverse [Fra14, p. 144].

²See, for instance, [vdW66, §3.1] and [ZS75, I.§5].

³Examples include [EGA I, 0.(1.0.1)], [Lan65, II.§1], [Wei67, p. XIV], and [AM69, p. 1].

⁴Compare [Jac51, p. 49] with [Jac85, p. 86], or [BML53, p. 370] with [BML65, p. 346], or [Bou58, I.§8.1] with [Bou70, I.§8.1].

⁵Some good reasons for requiring a 1 are explained in [Con13].

Theorem. A binary operation extends to a totally associative product if and only if it is associative⁶ and admits an identity element.

What?! Where did that identity element come from? The definition of totally associative implies the equations

$$(abc)d = abcd$$
$$(ab)c = abc$$
$$(a)b = ab$$
$$(a)a = a.$$

The last equation, which holds for any a, shows that the empty product () is a left identity. Similarly, () is a right identity, so () is an identity element.

Therefore it is natural to require rings to have a 1. But occasionally one does encounter structures that satisfy all the axioms of a ring except for the existence of a 1. What should they be called? Happily, there is an apt answer, suggested by Louis Rowen: rng!⁷ As our reasoning explains and as Rowen's terminology suggests, it is better to think of a rng as a ring with something missing than to think of a ring with 1 as having something extra.

3. Counterarguments

Here we mention some arguments for *not* requiring a 1, in order to rebut them.

- "Algebras should be rings, but Lie algebras usually do not have a 1."

 Lie algebras are usually not associative either. We require a 1 only in the presence of associativity. It is accepted nearly universally that ring multiplication should be associative, so when the word "algebra" is used in a sense broad enough to include Lie algebras, it is understood that algebras have no reason to be rings.
- "An infinite direct sum of nonzero rings does not have a 1."

 Direct sums are typically defined for objects like vector so
 - Direct sums are typically defined for objects like vector spaces and abelian groups, for which the set of homomorphisms between two given objects is an abelian group, for which cokernels exist, and so on. Rings fail to have these properties, whether or not a 1 is required. So it is strange even to speak of a direct sum of rings. Category theory explains that the natural notion for rings is the direct *product*. Each ring may be viewed as an *ideal* in the direct product; then their direct sum is an ideal too.
- "If a 1 is required, then function spaces like the space $C_c(\mathbb{R})$ of compactly supported continuous functions $f: \mathbb{R} \to \mathbb{R}$ will be disqualified."

This is perhaps the hardest to rebut, given the importance of function spaces. But many such spaces are ideals in a natural ring (e.g., $C_c(\mathbb{R})$ is an ideal in the ring $C(\mathbb{R})$ of all continuous functions $f: \mathbb{R} \to \mathbb{R}$), and fail to include the 1 only because of some condition imposed on their elements. So one can say that they, like the direct sums above and like the rng of even integers, deserve to be ousted from the fellowship of the ring. In any case, however, these function spaces still qualify as \mathbb{R} -algebras.

⁶I.e., associative in the usual sense, on triples: (ab)c = a(bc) for all $a, b, c \in A$.

⁷See [Jac85, p. 155]. Bourbaki introduced a different pejorative for the same concept: pseudo-ring [Bou70, I.§8.1]. Keith Conrad observes in [Con13, Appendix A] that the usual definition of (associative) ℤ-algebra gives the same notion and that hence this terminology could be used.

4. Final comments

Once the role of the empty product is acknowledged, other definitions that seemed arbitrary become natural. A ring homomorphism $A \to B$ should respect finite products, so in particular it should map the empty product 1_A to the empty product 1_B . A subring should be closed under finite products, so it should contain the empty product 1. An ideal is prime if and only if its complement is closed under finite products; in this case, 1 is in the complement; this explains why the unit ideal (1) in a ring is never considered to be prime.

The reasoning in Section 2 involved only one binary operation, so it explains also why monoids are more natural than semigroups.⁸ Similar reasoning explains why the axioms for a category require not only compositions of two morphisms but also identity morphisms: given objects A_0, \ldots, A_n and a chain of morphisms

$$A_0 \stackrel{f_1}{\to} A_1 \stackrel{f_2}{\to} \cdots \stackrel{f_n}{\to} A_n,$$

one wants to be able to form the composition, even if n = 0.

It would be ridiculous to introduce the definition of ring to beginners in terms of totally associative products. But it is nice to understand why certain definitions should be favored over others.

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⁸A semigroup is a set with an associative binary operation, and a monoid is a semigroup with a 1.

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