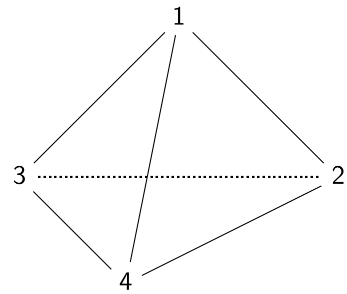
Symmetries of the tetrahedron. Math 113 Summer 2014.

Let Δ denote the tetrahedron sitting in $\mathbb{R}^3$, centred at the origin, with vertices labelled $1, 2, 3, 4$.



Let T denote the set of symmetries of Δ. There are two distinct types of symmetries in T - **reflections** and **rotations**. We will denote a symmetry s in T as a list (i_1, i_2, i_3, i_4), with $i_j \in \{1, 2, 3, 4\}$, and where i_j is the vertex position that vertex j is sent to via s. For example, if r_1 is the symmetry that fixes vertex 1 and rotates the plane containing the remaining three vertices by $2\pi/3$ counterclockwise (viewed from vertex 1), then we write $(1, 3, 4, 2)$.

This is the list of all 24 elements in T, with the above notation:

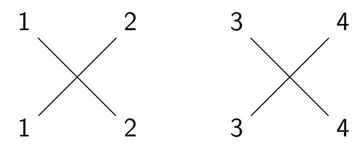
$(1,2,3,4)$	identity
$(1,3,4,2)$	rotation (fix vertex 1)
$(1,4,2,3)$	rotation (fix vertex 1)
$(4,2,1,3)$	rotation (fix vertex 2)
$(3,2,4,1)$	rotation (fix vertex 2)
$(2,4,3,1)$	rotation (fix vertex 3)
$(4,1,3,2)$	rotation (fix vertex 3)
$(3,1,2,4)$	rotation (fix vertex 4)
$(2,3,1,4)$	rotation (fix vertex 4)
$(2,1,4,3)$	rotation
$(3,4,1,2)$	rotation
$(4,3,2,1)$	rotation
$(1,2,4,3)$	reflection
$(1,3,2,4)$	reflection
$(2,1,3,4)$	reflection
$(1,4,3,2)$	reflection
$(4,1,2,3)$	reflection
$(2,4,1,3)$	reflection
$(3,1,4,2)$	reflection
$(3,2,1,4)$	reflection
$(2,3,4,1)$	reflection
$(4,3,1,2)$	reflection
$(4,2,3,1)$	reflection
$(3,4,2,1)$	reflection

It is a fact - that you shall prove later in this course - that each of these symmetries is a product of the reflections $(2,1,3,4)$, $(1,3,2,4)$, $(1,2,4,3)$. For example, the rotation $(1,3,4,2) = (1,3,2,4) * (1,2,4,3)$.

Let W_4 be the set of wiring diagrams on four vertices. Define

$$f : T \to W_4 \; ; \; (i_1, i_2, i_3, i_4) \mapsto X_{i_1 i_2 i_3 i_4}$$

where $X_{i_1 i_2 i_3 i_4}$ is the wiring diagram that has a line between j and i_j: for example, X_{2143} is the wiring diagram



f **is surjective**: let $X \in W_4$ be a wiring diagram. Let (i_1, i_2, i_3, i_4) be the list of endpoints of lines in X - so that i_1 is the endpoint of line starting at 1 etc. Then, $f(i_1, i_2, i_3, i_4) = X$, so that f is surjective. Hence, because f is surjective and $|T| = |W_4|$, f is bijective.

You can check that

$$(2,1,3,4) * (1,3,2,4) = (2,3,1,4),\ (2,3,1,4) * (2,1,3,4) = (3,2,1,4),$$

$$(2,3,1,4) * (2,3,1,4) = (3,1,2,4),\ (2,1,3,4) * (1,2,4,3) = (2,1,4,3)$$

$$(1,3,2,4) * (1,2,4,3) = (1,3,4,2).$$

It is straightforward to verify that $f(s * t) = f(s)f(t)$, for $s * t$ one of the five products appearing above. For example, $f(2,1,3,4) = X_{2134}$, $f(1,3,2,4) = X_{1324}$, and $X_{2134}X_{1324} = X_{2314}$