Sample Exam 2. Math 113 Summer 2014.

These problems are practice for the second exam, on rings and fields.

1. True or False

   (a) The canonical homomorphism $\pi\colon R \to R/I$ is surjective.

   (b) Every homomorphism of rings is injective.

   (c) The element $\overline{x}$ is a unit in $\mathbb{Q}[x]/(x^4+1)$.

   (d) There exists a homomorphism $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$.

   (e) If R is a unique factorization domain and I a proper ideal of R, then R/I is a unique factorization domain.

   (f) If $\sigma \in \mathrm{Gal}(L:K)$, and $\alpha \in L$ is a root of $f \in K[x]$, then $\sigma(\alpha)$ is a root of f.

   (g) If R and S are domains, then $R \times S$ is a domain.

   (h) Every algebraic field extension is finite.

   (i) $\mathbb{Q}(i - \sqrt{7}) = \mathbb{Q}(i, \sqrt{-7}+1)$.

   (j) The minimal polynomial of the extension $\mathbb{Q} \subset \mathbb{Q}(e^{2\pi i/3})$ is $x^3 - 1$.

   (k) If F is any field, there exists a homomorphism $F \to \mathbb{C}$.

   (l) If $K \subset L$ is a normal field extension of degree 4, then there exists exactly one intermediate subfield $F \neq K, L$.

   (m) The polynomial $3x^4 - 30x^2 + 10x + 15$ is irreducible over $\mathbb{Z}$.

   (n) If $f\colon R \to S$ is a surjective ring homomorphism, and $\mathfrak{m}$ a maximal ideal in S, then $f^{-1}(\mathfrak{m})$ is a maximal ideal in R.

   (o) There exists a homomorphism $\mathbb{Q}[x]/(x^2+2x+1) \to \mathbb{C}$.

2. (a) If R is a ring, say what it means for an element $r \in R$ to be irreducible.

   (b) Give an example of an irreducible polynomial of degree larger than 2 in the ring $\mathbb{Q}[x]$.

   (c) Let R be a domain and $I = (f)$ a nonzero ideal. Prove that if I is prime, then f is irreducible.

3. (a) Let $\alpha = \sqrt[3]{\sqrt{2}+\sqrt{3}}$, and consider the field extensions

   $$\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2}+\sqrt{3}) \subset \mathbb{Q}(\alpha) \subset \mathbb{Q}(i,\alpha) = K$$

   Given that $[K:\mathbb{Q}] = 24$, determine $[\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt{2}+\sqrt{3})]$. Justify your answer.

   (b) Let $\omega = \dfrac{-1+i\sqrt{3}}{2}$, a cube root of 1. Consider the extensions $\mathbb{Q} \subset \mathbb{Q}(\omega) \subset \mathbb{Q}(\sqrt[3]{7}, \omega)$. Let f and g be the automorphisms of $\mathbb{Q}(\sqrt[3]{7}, \omega)$ defined by

   $$f\colon \begin{cases} \sqrt[3]{7} \mapsto \omega\sqrt[3]{7} \\ \omega \mapsto \omega \end{cases} \qquad g\colon \begin{cases} \sqrt[3]{7} \mapsto \sqrt[3]{7} \\ \omega \mapsto \omega^2 \end{cases}$$

   Show that $f \in \mathrm{Gal}(\mathbb{Q}(\sqrt[3]{7}, \omega) : \mathbb{Q}(\omega))$.

   (c) Find an element $x \in \mathbb{Q}(\sqrt[3]{7}, \omega)$ such that $f(g(x)) \neq g(f(x))$.

(d) Using (d), and given that $[\mathbb{Q}(\sqrt[3]{7}, \omega) : \mathbb{Q}] = 6$, prove that $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{7}, \omega) : \mathbb{Q}) \cong S_3$.

(e) Prove that $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{7}, \omega) : \mathbb{Q}(\omega)) \cong \mathbb{Z}/3\mathbb{Z}$.

4. (a) For each of the following rings say, whether they are a field; domain; principal ideal domain; euclidean domain; unique factorization domain

   i. $\mathbb{Z}[x]$

   ii. $\mathbb{Q}[x]/(x^2 + x + 1)$

   iii. $\mathbb{C}[x, y]$

   (b) Define a principal ideal domain.

   (c) Prove that if R is a principal ideal domain and I a prime ideal of R, then R/I is a principal ideal domain.

   (d) Let $f : Z[x] \to \mathbb{Z}/2\mathbb{Z}$ be $f = g \circ h$, where $h \colon Z[x] \to \mathbb{Z}$ is the evaluation map at -1 and $g \colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the quotient map. Prove that $\ker f = (x + 1, x^2 + 1)$.

5. (a) State the (first) isomorphism theorem for rings.

   (b) Consider the map $\phi \colon \mathbb{C}[x, y] \to \mathbb{C}[y]$ given by $\phi(p(x, y)) = p(y^2, y^3)$. Compute $\phi(x^2 + xy + y^2)$.

   (c) Prove that $\mathrm{im}\,\phi = \mathbb{C}[y^2, y^3] \subset \mathbb{C}[y]$.

   (d) Prove that $\ker \phi$ is a prime ideal in $\mathbb{C}[x, y]$.

   (e) Is $\mathrm{im}\,\phi$ a unique factorization domain?