Homework 7 Math 113 Summer 2014.

Due Wednesday August 13th

Make sure to write your solutions to the following problems in complete English sentences. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make. Problems will be of varying difficulty, and do not appear in any order of difficulty. All rings are assumed commutative and with unity.

1. Compute the minimal polynomials and find the degree of the following simple extensions of $\mathbb{Q}$:
   (a) $\mathbb{Q}(\sqrt{-3})$
   (b) $\mathbb{Q}(\sqrt{3}+i)$
   (c) $\mathbb{Q}(\sqrt{2}-\sqrt{10})$
   (d) $\mathbb{Q}(e^{2\pi i/p})$, for p an odd prime.

2. Find a primitive element for each of the following extensions, then use this to find their minimal polynomial and degree:
   (a) $\mathbb{Q}(i,\sqrt{3})$
   (b) $\mathbb{Q}(\sqrt[4]{2},\sqrt{2})$
   (c) $\mathbb{Q}(\sqrt{2},\sqrt{10})$

3. Prove that, up to isomorphism, there are no finite extensions of $\mathbb{C}$ except $\mathbb{C}$ itself. In other words, if L is a finite extension of $\mathbb{C}$, then $L \cong \mathbb{C}$. Does there exist an algebraic extension of $\mathbb{C}$? Explain your answer.

4. Let K be the field obtained by adjoining all three cube roots of 2 to $\mathbb{Q}$. Show that K contains all cube roots of unity and compute its degree over $\mathbb{Q}$.

5. Suppose $f \in \mathbb{Q}[x]$, not necessarily irreducible.
   (a) Show that there is a smallest subfield of $\mathbb{C}$ over which f factors into linear factors. In other words, prove there exists a subfield K_f of $\mathbb{C}$ such that (i) f factors into linear factors in $K_f[x]$, and (ii) if L is any other subfield of $\mathbb{C}$ for which f factors into linear factors in $L[x]$, then $L \supseteq K_f$.
   (b) Taking $f = x^{12} - x^4 - 4x^3 + 4$, find K_f by writing at as $\mathbb{Q}(\alpha, \beta, ...)$, and compute the degree of K_f over $\mathbb{Q}$.

6. Define $\mathcal{A} = \{\alpha \in \mathbb{C} \mid \text{there exists } f \in \mathbb{Q}[x] \text{ such that } f(\alpha) = 0\}$ - this is the set of **algebraic numbers**. For example, $\sqrt{2} \in \mathcal{A}$ (since $f(\sqrt{2}) = 0$, where $f = x^2 - 2$), and $\sqrt{-2}+\sqrt{3} \in \mathcal{A}$ (since $g(\sqrt{-2}+\sqrt{3}) = 0$, where $g = x^4 - 2x^2 + 25$).
   (a) Show that $\mathbb{Q} \subset \mathcal{A}$.
   (b) Let $\mathbb{Q} \subset L$ be an algebraic extension of $\mathbb{Q}$. Prove that $L \subset \mathcal{A}$.
   (c) Prove that $\mathcal{A}$ is a field. Deduce that it is the largest algebraic extension of $\mathbb{Q}$ in $\mathbb{C}$.
   (d) Explain, using a single sentence, why $\mathcal{A} \neq \mathbb{C}$.[1]

[1] In fact, the algebraic integers are **countable**: this means that there is a bijection $\{1, 2, 3, ...\} \leftrightarrow \mathcal{A}$. The real numbers are **uncountable** so that there is no bijection $\{1, 2, 3, \dots, \} \leftrightarrow \mathbb{R}$; this means that there are *many* more real numbers than algebraic integers.

7. Let $K = \mathbb{Q}(i\sqrt[4]{2})$ and $L = \mathbb{Q}(i\sqrt[4]{2}, \sqrt{3})$, so that $\mathbb{Q} \subset K \subset L$.

   (a) Give an example of an embedding of K which is not an automorphism.

   (b) Give an example of an automorphism of L which does not fix K pointwise.

8. Consider the extension $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt[4]{2})$.

   (a) Prove that every automorphism of $\mathbb{Q}(\sqrt[4]{2})$ fixes $\mathbb{Q}(\sqrt{2})$ pointwise.

   (b) Deduce that $\mathrm{Gal}(\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}(\sqrt{2})) = \mathrm{Gal}(\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q})$.

   (c) Show that $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt[4]{2})$ is a normal extension, but $\mathbb{Q} \subset \mathbb{Q}(\sqrt[4]{2})$ is not.

   (d) Using (b), or otherwise, compute $\mathrm{Gal}(\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q})$.

9. Compute the Galois group of $\mathbb{Q} \subset \mathbb{Q}(i + \sqrt{2})$. List all intermediate subfields of the extension.

10. Compute the Galois group of $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$.