Homework 5 Math 113 Summer 2014.

Due Monday July 28th

Make sure to write your solutions to the following problems in complete English sentences. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make. Problems will be of varying difficulty, and do not appear in any order of difficulty.

1. Give examples of (commutative) subrings (with unity) R of $\mathbb{C}$ satisfying the following containments and non-containments. If it's not possible, explain why not. Note the symbol $\subset$ means "proper subset"[1]
   (a) $\mathbb{Q} \subset R$, $R \subset \mathbb{R}$
   (b) $\mathbb{Q} \subset \mathbb{R}$, $\mathbb{R} \not\subset R$.
   (c) $\mathbb{Z} \subset R$, $\mathbb{Q} \not\subset R$.
2. Prove that $\mathbb{Q}[\sqrt{2}]$ is a field. (Hint: think about division in $\mathbb{C}$!)
3. Let $\mathcal{L}(V)$ be the set of all linear maps from a vector space V to itself. It is a noncommutative ring with unity (the identity map), where the multiplication is composition of functions. If U is a subspace of V, prove that $\mathcal{L}(U)$ can be identified with a subring of $\mathcal{L}(V)$.
4. Let R be a ring (not necessarily commutative), and $\{e_1, \dots, e_k\}$ a set of idempotent elements such that $1 = e_1 + \dots + e_k$ and $e_i e_j = 0$ whenever $i \neq j$[2]. Prove that
$$R \cong e_1 R \times \cdots e_k R$$
5. Let R be a ring (not assumed commutative) in which every element is idempotent. Prove that R is in fact commutative, and satisfies $r + r = 0$ for every $r \in R$.
6. Decide which of the following sets are ideals in the given ring:
   (a) $\{p(x,y) \mid p(x,x) = 0\} \subset \mathbb{C}[x,y]$
   (b) $\{p(x,y) \mid p(x,y) = p(y,x)\} \subset \mathbb{C}[x,y]$
   (c) $\{p(x) \mid p \text{ has no real roots}\} \subset \mathbb{C}[x]$
7. Let R be a commutative ring with unity.
   (a) Prove that $\mathrm{Hom}(\mathbb{Z}, R)$ contains one element.
   (b) Give an example of a commutative ring with unity R such that $\mathrm{Hom}(R, \mathbb{Z}) = \varnothing$.
   (c) Give en example of a commutative ring with unity R such that $\mathrm{Hom}(R, \mathbb{Z})$ is infinite.
8. (a) Let $R = \mathbb{C}[x,y]$, $I = (y - x)$. What ring is R/I isomorphic to?
   (b) Let $R = \mathbb{Z}[x]$, $I = (3, x)$. What ring is R/I isomorphic to?
9. Prove that every field has exactly two ideals. What are they?
10. In this problem you will give a precise meaning to the process of 'adjoining elements' described in the notes.

---

[1] For this problem, at least.

[2] Such a set is called an **orthogonal system of idempotents**.

(a) Prove that any subring $R \subset \mathbb{C}$ contains $\mathbb{Z}$.

(b) Suppose that $(R_i)_{i \in I}$ is a family of subrings of $\mathbb{C}$ (ie, some arbitrary collection of subrings of $\mathbb{C}$). Prove that $\bigcap_{i \in I} R_i$ is a subring of $\mathbb{C}$.

(c) Let $Y \subset \mathbb{C}$ be a subset. Prove that there exists a subring $R \subset \mathbb{C}$ with the following property: if $S \subset \mathbb{C}$ is a subring such that $Y \subset S$, then $R \subset S$. (*The subring R just determined is often denoted $\mathbb{Z}[Y]$*)

(d) Let $Y = \{\sqrt{2}\} \subset \mathbb{C}$. Prove that there is an isomorphism of rings

$$\mathbb{Z}[x]/(x^2 - 2) \cong \mathbb{Z}[Y].$$

(Hint: Give an explicit description of $\mathbb{Z}[Y]$)

(e) Let $Y = \{\pi\} \subset \mathbb{C}$. Prove that there is an isomorphism of rings

$$\mathbb{Z}[x] \cong \mathbb{Z}[Y].$$

You will need to use the following fact: there is no polynomial $f \in \mathbb{Z}[x]$ such that $f(\pi) = 0$. (This means that π is a **transcendental** number.; it is a surprisingly hard result to prove.)

11. Let R be a commutative ring with unity.

    (a) Let $(I_j)_{j \in A}$ be a family of ideals in R (ie, some arbitrary collection of ideals in R). Prove that $\bigcap_{j \in A} I_j$ is an ideal in R.

    (b) Let $X \subset R$ be an arbitrary subset. Prove that there exists a unique ideal $I \subset R$ with the following property: if J is an ideal and $X \subset J$, then $I \subset J$. (*We call the ideal I just determined* **the ideal generated by** X, *and denote it* $(X) \subset R$.)

    (c) Determine (X) for the following subsets $X \subset R$

    $$X = \{x - 1, x + 1\} \subset R = \mathbb{R}[x], \; X = \{x^2 + 1, x^2 - 1\} \subset R = \mathbb{C}[x].$$

12. Prove that if an element u in a commutative ring R with unity is not contained in any proper ideal, then u must be a unit.

13. Let $R = \mathbb{Z}/n\mathbb{Z}$, for some $n > 1$. Prove that $I \subset R$ is an ideal if and only if I is an abelian subgroup of $(R, +)$.