Homework 3 Math 113 Summer 2014.

Due Thursday July 10th

Make sure to write your solutions to the following problems in complete English sentences. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make. Problems will be of varying difficulty, and do not appear in any order of difficulty.

1. Let S_3 act on $\text{Sub}(S_3)$, the set of subgroups of S_3, by conjugation. Describe all the orbits of this action; calculate the stabilisers of $\langle(12)\rangle$ and $\langle(132)\rangle$.

2. Suppose that G is a finite group with the following property: let $H, K \subset G$ be subgroups such that $|H| = |K|$, then $H = K$. Prove that every subgroup in G is normal. (*Hint: Use the group action of G acting on* $\text{Sub}(G)$ *by conjugation*)

3. (a) Prove that if an element σ of S_n has cycle type $(n_1, \ldots, n_k)$, then its order is the least common multiple $\text{lcm}(n_1, \ldots, n_k)$ of $n_1, \ldots, n_k$.

   (b) Find the number of elements of each order in S_5.

4. Find three non-isomorphic abelian groups of order 8.

5. Let G be a group.

   (a) Let $g \in G$. Prove that the conjugation map $c_g \colon h \mapsto ghg^{-1}$ is an automorphism of G.

   (b) Prove that the map $c \colon G \to \text{Aut}G$ given by $g \mapsto c_g$ is a homomorphism, whose kernel is $Z(G)$, the centre of G .

6. Prove Cauchy's Theorem: Let G be a group of order n, and p a prime dividing n. Then there exists an element g in G of order p.

7. Determine (up to isomorphism) all *finite* groups which have exactly two conjugacy classes.

8. Are the permutations $(1354)(23)(245)$ and $(3452)(12)(123)$ conjugate in S_9?

9. Let G be a group of order 35 and let $x \in G$ satisfy $o(x) = 7$.

   (a) Using Sylow's Theorems give an explanation as to why such an x must exist.

   (b) Prove that $H = \langle x \rangle$ is normal in G.

   (c) Part (b) implies that G acts on H by conjugation. Prove that there exists a nontrivial element $h \in H$ such that $\text{Cent}_G(h) = G$.

10. (a) Let G be a group and H a Sylow p-subgroup. Prove that H is the *only* Sylow p-subgroup if and only if H is normal.

    (b) Let G be a group of order 63. Prove that G contains a cyclic normal subgroup.

11. (a) Prove that the centre of a p-group is nontrivial.

    (b) Let G be a group of order p^2. Show that $\text{Cent}_G(x) = G$, for every $x \in G$. Deduce that G is abelian.

    (c) Give an example of a non-abelian group of order p^3, for some prime p.

12. Let $G = \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$, where p is a prime. Using Sylow's Theorems prove that the following subgroups are conjugate in G

$$U_2(\mathbb{Z}/p\mathbb{Z}) \stackrel{def}{=} \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{Z}/p\mathbb{Z} \right\},\ U_2' \stackrel{def}{=} \left\{ \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \mid a \in \mathbb{Z}/p\mathbb{Z} \right\}.$$

13. Let G act on the set S, $x \in S$. Prove that

$$\mathrm{Stab}_G(g \cdot x) = g\mathrm{Stab}_G(x)g^{-1}.$$