Math 113, Summer 2014: **Exam 1**

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Thursday, July 17th

Attempt to answer all of the following FIVE questions. You DO NOT need to justify your response to the TRUE/FALSE problems.

1. This is a closed book exam. Please put away all your notes, textbooks, calculators and portable electronic devices and turn your mobile phones to 'silent' (non-vibrate) mode.

2. Explain your answers **CLEARLY** and **NEATLY** and in **COMPLETE ENGLISH SENTENCES**. State all theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and fully.

3. Correct answers without appropriate justification will be treated with great skepticism.

4. Write your name on this exam and any extra sheets you hand in.

Question 1:	/20
Question 2:	/20
Question 3:	/25
Question 4:	/30
Question 5:	/30
Total:	/125

Name: Solution

1. (20 points)

   (a) Let G be a group, $H \subset G$ a subset. Define what it means for H to be a **subgroup** of G.

   (b) Suppose that $H, K \subset G$ are subgroups such that $H \not\subset K$ and $K \not\subset H$. Prove that

   $$H \cup K = \{g \in G \mid g \in H \text{ or } g \in H\},$$

   is <u>not</u> a subgroup.

   (c) Let $H, K \subset G$ be subgroups. Define

   $$HK = \{hk \mid h \in H, k \in K\} \subset G.$$

   Give an example of two subgroups $H, K, \subset S_3$ such that HK is <u>not</u> a subgroup.

Solution:

(a) Let $g, h \in H$. Then, $gh^{-1} \in H$.

(b) As $H \not\subset$K there is some $h \in H$ such that $h \notin K$. Similarly, there is some $k \in K$ such that $k \notin H$. Then, if $H \cup K$ is a subgroup, we would require that $hk^{-1} \in H \cup K$. However, if $hk^{-1} \in H \cup K$ then either $hk^{-1} \in H$ or $hk^{-1} \in K$. If $hk^{-1} \in H$, then $hk^{-1} = h' \implies k^{-1} = h^{-1}h' \in H \implies k \in H$, which is absurd. Similarly, if $hk^{-1} \in K$, then we would get $h \in K$. Hence, it is not possible that $hk^{-1} \in H \cup K$, so that it is not a subgroup.

(c) Let $H = \{e, (12)\}, K = \{e, (23)\}$. Then,

$$HK = \{e, (12), (23), (123)\}.$$

Since $|HK| = 4$ and 4 does not divide 6, HK is not a subgroup, by Lagrange's Theorem.

2. True/False (20 points - 2 points each). No justification required.

   (a) If $f\colon G \to H$ is a homomorphism, and $H \cong G/\ker f$, then f is surjective.

   (b) The permutations (135)(2435) and (123)(456) are conjugate in S_6.

   (c) If H is a subgroup of G and $\mathrm{Norm}_G(H)$ is its normalizer, then H is normal in $\mathrm{Norm}_G(H)$.

   (d) If G is a p-group and N a normal subgroup of G, then G/N is a p-group.

   (e) If H is a normal subgroup of G, then for all $h \in H$, $\mathrm{Cent}_G(h) = G$.

   (f) Let G be a group acting on the set S and suppose that $|G| = 343 = 7^3$, $|S| = 9$. Then, S admits exactly three orbits.

   (g) If G, H are cyclic, then $G \times H$ cyclic.

   (h) There exists a injective homomorphism $\mathbb{Z}/5\mathbb{Z} \to D_{10}$.

   (i) Suppose $|G| = 8$. Then, it is possible that $|Z(G)| = 4$.

   (j) In S_5, the elements of order 4, together with the identity element, form a subgroup.

   Solution:

   (a) T
   (b) F
   (c) T
   (d) T
   (e) F
   (f) F
   (g) F
   (h) T
   (i) F
   (j) F

3. (25 points)

(a) Prove that $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/25\mathbb{Z}$.

(b) Let A be an abelian group of order 675. List all possible isomorphism classes of A.

(c) Assume further that A contains an element of order 225. List the possible isomorphism classes of A.

(d) Let $G = \mathbb{Z}/25\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Find subgroups $H, K \subset G$ of order 3 such that G/H and G/K are not isomorphic.

Solution:

(a) Since $5(\bar{i}, \bar{j}) = (\overline{5i}, \overline{5j}) = (\bar{0}, \bar{0})$, for any $(\bar{i}, \bar{j}) \in \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, the order of every element is no larger than 5. However, $\mathbb{Z}/25$ contains an element of order 25 (namely $\bar{1}$), so that these two groups can't be isomorphic.

(b) Since $675 = 3^3.5^2$, we use the algorithm described in Lecture 13, with $p_1 = 3, p_2 = 5$, $n_1 = 3, n_2 = 2$. We need all partitions of 3 and 2: they are

$$(3), (2,1), (1,1,1), \quad (2), (1,1).$$

Each isomorphism class corresponds to a pair of partitions (λ, μ), where λ is a partition of 3, μ is a partition of 2. We find

$$((3),(2)): \quad \mathbb{Z}/675\mathbb{Z},$$

$$((2,1),(2)): \quad \mathbb{Z}/225\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z},$$
$$((1,1,1),(2)): \quad \mathbb{Z}/75\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z},$$
$$((3),(1,1)): \quad \mathbb{Z}/135\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z},$$
$$((2,1),(1,1)): \quad \mathbb{Z}/45\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z},$$
$$((1,1,1),(1,1)): \quad \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

(c) Since the only groups above containing an element of order 225 are the first and second listed, we obtain

$$\mathbb{Z}/675\mathbb{Z}, \quad \mathbb{Z}/225\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

(d) Note that the subsets

$$H = \{(\bar{0}, \bar{0}, x) \mid x \in \mathbb{Z}/3\mathbb{Z}\} \subset G, \quad K = \{(\bar{0}, \overline{3i}, \bar{0}) \mid i = 0, 1, 2\} \subset G,$$

are subgroups of order 3. A theorem from class states that G/H is isomorphic to $\mathbb{Z}/25\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$ (Lecture 10); hence G/H contains an element of order 9, namely $(0, \bar{1})$.

Now, let $(\bar{i}, \bar{j}, \bar{k}) + K \in G/K$. Notice that

$$3\left((\bar{i}, \bar{j}, \bar{k}) + K\right) = (\overline{3i}, \overline{3j}, \overline{3k}) + K = (\overline{3i}, \bar{0}, \bar{0}) + K$$

since $(\overline{3i}, \overline{3j}, \overline{3k}) - (\overline{3i}, \bar{0}, \bar{0}) = (\bar{0}, \overline{3j}, \overline{3k}) = (\bar{0}, \overline{3i}, \bar{0}) \in K$. In particular, for any r,

$$3r\left((\bar{i}, \bar{j}, \bar{k}) + K\right) = (\overline{3ri}, \bar{0}, \bar{0}) + K.$$

Thus, if G/K contains an element of order 9 then we require $\overline{9i} = \bar{0}$, so that $9i$ is divisible by 25. This only holds if i is divisible by 25, so that $\bar{i} = \bar{0}$. But then, we have that

$$3\left((\bar{0}, \bar{j}, \bar{k}) + K\right) = (\bar{0}, \bar{0}, \bar{0}) + K,$$

so that no element can have order 9 (as we've shown that such an element must have order ≤ 3).

4. (30 points)

   (a) Give the definition of a Sylow p-subgroup of a group G.

   (b) Let G be a group of order 45. Prove that there is exactly one Sylow 3-subgroup H and exactly one Sylow 5-subgroup K.

   (c) Let G be a group of order $p^r m$ with $\gcd(p, m) = 1$, $m > 1$, and $|G| > [G : \mathrm{Norm}_G(H)]!$, for $H \in \mathrm{Syl}_p$. Prove that G is not simple[1].

Solution:

(a) Let $|G| = p^r.m$, where $\gcd(p, m) = 1$. Then, a Sylow p-subgroup is a subgroup of G of order p^r.

(b) As $45 = 3^2.5$, there exists a Sylow 3-subgroup and a Sylow 5-subgroup, by Sylow's Theorems. The number of Sylow 3-subgroups divides 5 and is equivalent to $1 \mod 3$. Hence, there is only one of them. Similarly, the number of Sylow 5-subgroups divides 9 and is equivalent to $1 \mod 5$. Hence, there is only one of them.

(c) The number of Sylow p-subgroups is exactly $[G : \mathrm{Norm}_G(H)]$, for $H \in \mathrm{Syl}_p$. Since G acts nontrivially on Syl_p by conjugation (as all Sylow p-subgroups are conjugate to each other), there is a homomorphism

$$f : G \to \mathrm{Perm}(\mathrm{Syl}_p).$$

We have that $|\mathrm{Perm}(\mathrm{Syl}_p))| = [G : \mathrm{Norm}_G(H)]!$, so that f cannot be injective, using the condition given on $|G|$, and that $|G| = |\mathrm{im} f||\ker f|$. Hence, $|\ker f| > 1$. As f is nontrivial $|\ker f| < |G|$. Hence, $\ker f$ is a proper nontrivial normal subgroup of G.

[1] Recall that a group is **simple** if it has no proper nontrivial normal subgroups.

5. (30 points)

(a) Let $K = \{e, s, sr^2, r^2\} \subset D_8$. The subgroup $\langle s \rangle$ acts on K by conjugation. How many orbits are there for this action?

Let G be a group of order 99. There exists exactly one Sylow 3-subgroup H and exactly one Sylow 11-subgroup K. (You **do not** have to show this!)

(b) Explain why the following action is well-defined:

$$K \times H \to H \; ; \; (k, h) \mapsto k \cdot h = khk^{-1}.$$

(Hint: why is $k \cdot h \in H$?)

(c) Prove that the action described in part (b) is trivial.[2]

(d) Show that there exists $x \in K$ such that $G/H = \{x^i H \mid i = 0, \dots 10\}$. Deduce that $K \subset Z(G)$.

Solution:

(a) Recall that $s^{-1} = s \in D_8$ and $sr = r^{-1}s$. We have

$$ses = s, \; sss = s, \; sr^2 s = r^2, \; s(sr^2)s = r^2 s = sr^2.$$

Hence, $\mathcal{O}_x = \{x\}$, for any $x \in K$. Thus, there are four orbits.

(b) Since H is normal we know that $ghg^{-1} \in H$, for any $g \in G$. In particular, $k \cdot h = khk^{-1} \in H$, for any $k \in K \subset G$.

(c) We have a group K, with $|K| = 11$, acting on a set H, with $|H| = 9$. The Orbit-Stabiliser Theorem tells us that the size of any orbit must divide 11. As $|H| < 11$, the only possible size of an orbit is 1. Hence, for any $h \in H, \mathcal{O}_h = \{h\}$, so that $k \cdot h = h$, for any $k \in K, h \in H$.

(d) As $|K| = 11$, K is a cyclic group, generated by an element $x \in K$. We claim that the left cosets of H, $x^i H, x^j H$, are distinct, whenever $i \neq j$ and $0 \leq i, j \leq 10$. Suppose that $x^i H = x^j H$, with $i \neq j$, $i, j \in \{0, \dots, 10\}$. Assume, without loss of generality, that $i > j$. Then, $x^i = x^j h$, for some $h \in H$. Hence, $x^{i-j} = h$. Thus, $h \in H \cap K \subset K$. As $H \cap K \subset H$ then $|H \cap K| \leq |H| = 9$, and since $H \cap K \subset K$ is a subgroup it has order dividing 11. The only possibility is that $|H \cap K| = 1 \implies H \cap K = \{e\}$. Hence, $x^{i-j} = e \implies i = j$, which is absurd. Thus, the cosets described above are distinct. As $|G/H| = |G|/|H| = 99/9 = 11$, then we must have

$$G/H = \{x^i H \mid i = 0, \dots, 10\}.$$

Now, we have just shown that we can write any $g \in G$ as $g = x^i h$, for some $i = 0, \dots, 10$ and $h \in H$. Let $x^j \in K$. Then,

$$x^j g = x^j x^i h = x^{j+i} h = h x^{i+j} = h x^i x^j = g x^j,$$

where we have used that, for any $h \in H$, $h = x^m \cdot h = x^m h x^{-m} \implies x^m h = h x^m$. Hence, $x^j \in Z(G)$, for any j, so that $K \subset Z(G)$.

[2] An action of G on S is **trivial** if $g \cdot s = s$, for every $s \in S$, $g \in G$.