Worksheet 7/9. Math 113 Summer 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Problems with an asterisk (*) should be more challenging than the rest.

1. (a) Describe the quotient group $\mathbb{R}^\times/P$, where P is the subgroup of positive real numbers.

   (b) Describe the quotient group $\mathbb{C}^\times/P$, where P is the set of positive real numbers (here thought of as a subgroup of $\mathbb{C}^\times$.

2. For each of the following pairs of a group G and a normal subgroup H of G, match the quotient G/H with an isomorphic group from the following list: $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\left\{\begin{bmatrix} \pm1 & 0 \\ 0 & \pm1 \end{bmatrix}\right\}$ (under multiplication)

   (a) $G = S_3$, $H = \{e, (123), (132)\}$.

   (b) $G = \mathcal{Q}$, $H = \{\pm 1\}$.

   (c) $G = \mathbb{Z}/8\mathbb{Z}$, $H = <\overline{4}>$

   (d) $G = \mathbb{Z}/8\mathbb{Z}$, $H = <\overline{2}>$

3. Verify that the remaining two axioms hold for G/N to be a group; namely prove that each element gN has an inverse, and prove associativity of the group law.

4. Let N be a normal subgroup of G, and suppose gN has order n in the quotient group G/N. Prove that $g^m \in N$ if and only if $n|m$.

5. * Prove the following theorem by following the steps below: If H is a normal subgroup of G whose index is coprime to its order, then H is the *only* subgroup of its size in G.

   (a) If $f\colon G \to H$ is a homomorphism, and $\gcd(|G|, |H|) = 1$, then f is trivial.

   (b) Let $\pi\colon G \to G/N$ be the canonical homomorphism. If K is a subgroup of G such that $\pi(k) = e$ for all $k \in K$, then $K \subset N$.

   (c) Deduce the theorem.

6. * Let $T \subset \mathrm{GL}_3(\mathbb{R})$ be the subgroup of diagonal matrices.

   (a) Show that $\mathrm{Norm}_{\mathrm{GL}_3(\mathbb{R})}(T)$ consists of those 3×3 matrice that contain a single nonzero entry in each row and column.

   (b) Prove that $S_3 \cong \mathrm{Norm}_{\mathrm{GL}_3(\mathbb{R})}(T)/T$.