Worksheet 7/7. Math 113 Summer 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Problems with an asterisk (*) should be more challenging than the rest.

1. (a) Let $\sigma \in S_4$ be such that $o(\sigma) = 2$. What are the possible cycle types π_σ?
   (b) Determine $|\mathrm{Cent}_{S_4}((12))|$ and $|\mathrm{Cent}_{S_4}((12)(34))|$.
   (c) Determine the number of elements in S_4 that have order two.
   (d) Do the same problem but in S_5.

2. Prove Fermat's Little Theorem - *if $a \in \mathbb{Z}$ and p is prime, then* $\overline{a^p} = \overline{a} \in \mathbb{Z}/p\mathbb{Z}$ - using group actions, as follows: consider the set

$$S = \{(x_0, \dots, x_{p-1}) \mid x_i \in \{1, \dots, a\}\}.$$

   Prove that the 'cyclic shift' action

$$\overline{i} \cdot (x_0, \dots, x_{p-1}) = (x_{\overline{i}}, x_{\overline{i+1}}, \dots, x_{\overline{i+p-1}}),$$

   defines a group action of $\mathbb{Z}/p\mathbb{Z}$ on S. For example, $\overline{2} \cdot (1, 2, 3, 4, 5) = (3, 4, 5, 1, 2)$. Use the fixed point congruence to obtain the result.

3. Let S_3 act on the collection of all polynomials in three variables with real coefficients as follows

$$\sigma \cdot f(x_1, x_2, x_3) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

   For example,

$$(12) \cdot (x_1 - 2x_2^2) = x_2 - 2x_1^2$$
$$(123) \cdot (x_1x_2 + x_2x_3 + x_1x_3) = x_2x_3 + x_3x_1 + x_2x_1.$$

   (a) Explain why the number of polynomials that can be obtained from a given polynomial f by permuting the variables is a divisor of 3!. (*This is the original problem that Lagrange solved, providing the first glimpse of the Theorem that bears his name*)
   (b) A polynomial f (in three variables) is called **symmetric** if it is a fixed point of the above action. Determine the homogeneous symmetric polynomials (in three variables) with total degree $1, 2$.[1]
   (c) * The **elementary symmetric polynomials of degree** $1, 2$ are

$$e_1 = x_1 + x_2 + x_3, \quad e_2 = x_1x_2 + x_1x_3 + x_2x_3.$$

   Show that e_1, e_2 are symmetric and that there is a polynomial $g(y, z)$ such that

$$g(e_1, e_2) = x_1^2 + x_2^2 + x_3^2.$$

4. * Let G be a finite group, $H \subset G$ a p-subgroup, for some prime p. Suppose that p divides $[G : H]$. Then, p divides $[\mathrm{Norm}_G(H) : H]$; in particular, $\mathrm{Norm}_G(H) \neq H$. (*Hint: think about H acting on the set of left cosets G/H*)

[1] The total degree of a polynomial f is the degree of the polynomial obtained by replacing x_1, x_2, x_3 by the variable X; call this polynomial $f(X)$. For example, the total degree of $f = x_1x_2 + x_3x_1^2$ is 3, since $f(X) = X^2 + X^3$. A polynomial is homogeneous if $f(X)$ contains terms of equal degree; eg. $f = x_1x_2 + x_3x_1^2$ is not homogeneous, while $g = x_1x_2x_3 + x_2^3$ is homogeneous.