Worksheet July 30. Math 113 Summer 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Problems with an asterisk (*) should be more challenging than the rest. <u>*All rings R are assumed commutative and with unity, unless explicitly stated otherwise*</u>*.*

1. (a) Show that $f = x^4 + 2x + 2 \in \mathbb{Q}[x]$ is irreducible. (Hint: Eis...)
   (b) Show that $\mathbb{Q}[x]/(f)$ is an integral domain.
   (c) Show that $\overline{x} \in \mathbb{Q}[x]/(f)$ is a unit by finding an explicit inverse. Conclude that $\overline{x+1} \in \mathbb{Q}[x]/(f)$ is a unit, and find an explicit inverse.
   (d) What theoretical reason implies that these elements must be units?

2. Prove that the following rings are not UFDs; which of them are domains?
   (a) $\mathbb{Q}[x]/(x^3 - 8)$
   (b) $\mathbb{Z}[x]/(x^4 + 2x + 2)$
   (c) $\mathbb{Q}[x^2, x^3] \subset \mathbb{Q}[x]$

3. Suppose that $R[x]$ is a PID. Prove that R is a field.

4. Provide a counterexample to the following statement: if $f : R \to S$ is a homomorphism of rings and $\mathfrak{m} \subset S$ is maximal, then $f^{-1}(\mathfrak{m}) \subset R$ is maximal.

5. Let $f : S \to k$ be a homomorphism of rings, with k a field. Prove:
   (a) If f is not injective then S is not a field.
   (b) $S/\ker f$ is a domain.

6. * Let D be an integer ≥ 1, $R = \mathbb{Z}[\sqrt{-D}] \subset \mathbb{C}$.
   (a) Prove that complex conjugation in $\mathbb{C}$ restricts to give an automorphism of R.
   (b) Prove that if $D \geq 2$ then the only units are± 1.
   (c) Prove that $2, 3, 2 \pm \sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ are irreducible.
   (d) Can you find another D such that $\mathbb{Z}[\sqrt{-D}]$ is **not** a UFD?