Worksheet July 24. Math 113 Summer 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Problems with an asterisk (*) should be more challenging than the rest. *All rings R are assumed commutative and with unity, unless explicitly stated otherwise.*

1. Find a subring $R \subset \mathbb{C}$ such that the following quotient rings are isomorphic to R. Try to prove that R is isomorphic to the given quotient by defining an explicit isomorphism.

   (a) $\mathbb{Z}[x]/(x^2-3)$

   (b) $\mathbb{Q}[x]/(x-5)$

   (c) $\mathbb{Z}[x]/(x^2+1)$

   (d) $\mathbb{Q}[x]/(x^3-2)$

   (e) $\mathbb{R}[x]/(x^2+2x+2)$

   (f) *(little bit harder) $\mathbb{Q}[x,y]/(x^2-2, y^2-3)$

2. Let R be a ring, $I \subset R$ an ideal. Show that $\pi : R \to R/I$; $p \mapsto p + I$ is a ring homomorphism.

3. (a) Let $I = (x+y, xy) \subset \mathbb{R}[x,y]$. Show that $\mathbb{R}[x,y]/(x+y, xy) = \{a\overline{1} + b\overline{x} \mid a, b \in \mathbb{R}\}$.

   (b) Show that $f : \mathbb{R}[x]/(x^2) \to \mathbb{R}[x,y]/(x+y, xy)$; $p + (x^2) \mapsto p + (x+y, xy)$, is an isomorphism of rings. (You can assume that f is well-defined)

4. * In this problem you will see how 'reducing a polynomial mod p' can help determine whether roots exist or not.

   (a) Let p be a prime. Show that $r_p : \mathbb{Z}[x] \to \mathbb{Z}/p\mathbb{Z}[x]$; $\sum_{i=0}^n a_i x^i \mapsto \sum_{i=0}^n \overline{a_i} x^i$ is a ring homomorphism.

   (b) We say that a nonzero polynomial $f \in R[x]$ is **reducible** if $f = gh$, for some $g, h \in R[x]$, $g, h \in R[x]$ non-units; otherwise, we say that f is **irreducible**. Show that x^2-2 and x^2+1 are irreducible in $\mathbb{Z}[x]$ (Recall: the units are $\{\pm 1\} \subset \mathbb{Z}[x]$). (Hint: if it were reducible what would the divisors have to be?)

   (c) Suppose that $f \in \mathbb{Z}[x]$ is reducible. Show that if $r_p(f) \neq 0 \in \mathbb{Z}/p\mathbb{Z}[x]$, then it is reducible in $\mathbb{Z}/p\mathbb{Z}[x]$.

   (d) Let $p = 3$. Show that $\overline{f} = x^3 - x - \overline{1} \in \mathbb{Z}/3\mathbb{Z}[x]$ does not possess any roots in $\mathbb{Z}/3\mathbb{Z}$. Conclude that $\overline{f}$ is irreducible and deduce that $f = x^3 - x - 1 \in \mathbb{Z}[x]$ is irreducible; hence, it has no roots in $\mathbb{Q}$. (You will need the following: if $\deg g = 3$, $g \in \mathbb{Z}[x]$ then g is reducible if and only if g admits a root in $\mathbb{Q}$. Better still, prove this!)