Worksheet July 23. Math 113 Summer 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Problems with an asterisk (*) should be more challenging than the rest. *All rings R are assumed commutative and with unity, unless explicitly stated otherwise.*

1. Prove proposition 3.1.3 from class, namely: Let $f : R \to S$ be a homomorphism of rings. Prove that f is injective if and only if $\ker f = \{0_R\}$; prove that f is surjective if and only if $\operatorname{im} f = S$.

2. Calculate the kernels of the following maps, by giving a set of generators for each kernel.

   (a) $\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$; $\overline{n} \mapsto \overline{n}$ (the first congruence class is mod p^2; the second, mod p)

   (b) $\mathrm{Ev}_2 : \mathbb{Q}[x] \to \mathbb{Q}$; $p(x) \mapsto p(2)$

   (c) $\mathbb{R}[x] \to \mathbb{C}$; $p(x) \mapsto p(i)$ (hint: recall the factorization of real polynomials into linear and irreducible quadratic factors)

3. Which of the following subsets are ideals?

   (a) $\mathbb{Z}[x^2] \subset \mathbb{Z}[x]$

   (b) $R \subset S$, where R is any proper subring.

   (c) $\{p(x) \in \mathbb{C}[x] \mid p(i) = 0\} \subset \mathbb{C}[x]$

4. If $I = (x)$ and $J = (x + i)$ in $\mathbb{C}[x]$, determine (in other words, find generators for) $I \cap J$ and $I + J$.

5. * Recall that $\mathbb{C}[[x]]$ is the ring of formal power series over $\mathbb{C}$, whose elements are infinite series $\sum_{n=0}^{\infty} a_n x^n$, where we don't care whether the series converges or not.

   (a) Prove that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a unit in $\mathbb{C}[[x]]$ if and only if $a_0 \neq 0$ (this was on Monday's worksheet as well, so if you solved it then, move on to the next part)

   (b) Suppose I is a proper ideal of $\mathbb{C}[[x]]$. Then I cannot contain a unit. What is the largest possible such ideal I?

   (c) Use this to determine all ideals in $\mathbb{C}[[x]]$.

   (d) $\mathbb{C}[x] \subset \mathbb{C}[[x]]$ is a subring. But $\mathbb{C}[x]$ has many more ideals than $\mathbb{C}[[x]]$, even though $\mathbb{C}[[x]]$ is much much larger! Give some examples of ideals of $\mathbb{C}[x]$ which are no longer ideals when we consider them as subsets of $\mathbb{C}[[x]]$.