Worksheet July 22. Math 113 Summer 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Problems with an asterisk (*) should be more challenging than the rest. All rings R are assumed commutative and with unity, unless explicitly stated otherwise.

1. (a) Show that if $a \in \mathbb{Z}$ then $\mathbb{Z}[a] = \mathbb{Z}$.
   (b) Let $a \in \mathbb{C}$. Prove that if $R \subset \mathbb{C}$ is a subring such that $a \in R$ then $\mathbb{Z}[a] \subseteq R$.
   (c) Show that $\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{3}] \subset \mathbb{Z}[\frac{1}{6}]$.
2. (a) Prove that $\mathrm{id}_R : R \to R \, ; \, x \mapsto x$ is a ring homomorphism.
   (b) Let $f : R \to S$ and $g : S \to T$ be ring homomorphisms. Prove that $g \circ f : R \to T$ is a ring homomorphism.
3. Which of the following functions are ring homomorphisms?
   (a) $f : \mathbb{Q} \to \mathbb{Q}[x] \, ; \, a \mapsto ax$,
   (b) $R \to R \times S \, ; \, r \mapsto (r, 0)$,
   (c) $R \times S \to R \, ; \, (r, s) \mapsto r$,
   (d) $R \to R \times S \, ; \, r \mapsto (r, 1)$,
   (e) $\mathbb{Z}[i] \to \mathbb{Z}[i] \, ; \, a + bi \mapsto a - bi$
4. Let $f : \mathbb{Z} \to R$ be a homomorphism (in fact, there is only *one* such homomorphism!). Suppose that $2 = 1 + 1 \in R$ is a unit.
   (a) Suppose that $2 = 1 + 1 \in R$ is a unit. Define a ring homomorphism $g : \mathbb{Z}[\frac{1}{2}] \to R$ such that $g(n) = f(n)$, for every $n \in \mathbb{Z}$. Why is this last condition redundant?
   (b) Suppose that $f(2) \in R$ is a unit. Define a ring homomorphism $\mathbb{Z}[\frac{1}{2}] \to R$. Explain why this problem is the same as that considered in (a).
   (c) Let $g' : \mathbb{Z}[\frac{1}{2}] \to R$ be a ring homomorphism. Prove that $g' = g$.
5. Prove that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is isomorphic as a ring to $\mathbb{Z}/6\mathbb{Z}$. (Hint: it's perhaps easier to define a homomorphism $f : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and check it is bijective)
6. * Let R be a subring of the field K (eg $R = \mathbb{Z}, K = \mathbb{Q}$).
   (a) Prove that R does not possess any zerodivisors.
   (b) Let $A = \{a_1, \dots, a_k\} \subset R$ be a subset of nonzero elements. Suppose that $f : R \to S$ is a ring homomorphism such that $\{f(a_1), \dots, f(a_k)\} \subset S$ consists of units in S. Prove that there exists a **unique** homomorphism $g : R[A^{-1}] \to S$, such that $g(r) = f(r)$, for every $r \in R$. Here $A^{-1} = \{a_1^{-1}, \dots, a_k^{-1}\} \subset K$ is the set of inverses (in K) of the elements of A, and we are considering $R[A^{-1}] \subset K$.