Worksheet 7/2. Math 113 Summer 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework.

1. True or False (give a brief reason for each true statement, or cite a result from lecture; think of a counterexample for each false statement).

   (a) When a group G acts on a set X, all the orbits of the action have the same size.

   (b) The center of any group G is a normal subgroup of G.

   (c) If an element $g \in G$ satisfies $g^n = e$, then the order of g is n.

   (d) If a group has n elements, and there are exactly n conjugacy classes, then the group must be abelian.

2. In S_5, using the cycle notation from class, calculate the following (meaning write the given product as a product of *disjoint* cycles). Remember that they're functions, so you work from right to left (as in composition of functions). Are these elements conjugate in S_5?

   (a) $(132)(2435) =$

   (b) $(12)(123)(245) =$

3. Prove that a subgroup H of G is normal[1] if and only if $gH = Hg$ for all $g \in G$.

4. Prove the class equation from lecture, namely, if G is a finite group, and $\{e, g_1, \dots, g_k\}$ are representatives for each of the distinct conjugacy classes in G, then

$$|G| = 1 + \sum_{i=1}^{k} |C(g_i)|$$

5. * Let $G = SL_2(\mathbb{Z}/3\mathbb{Z})$, the **special linear group over** $\mathbb{Z}/3\mathbb{Z}$, which is defined as $\{A \in M_2(\mathbb{Z}/3\mathbb{Z}) \mid \det A = 1\}$, and let

$$H = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \middle| a \in \mathbb{Z}/3\mathbb{Z} \right\}.$$

   Prove that the normalizer of H in G is a cyclic group of order 6.

[1] using our definition of normal - many books actually use this condition as the definition of normal, in which case there would be nothing to prove!