Worksheet August 4th. Math 113 Summer 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Problems with an asterisk (*) should be more challenging than the rest.

1. Determine the degrees of the following field extensions by writing down a basis
   (a) $\mathbb{Q} \subset \mathbb{Q}(i\sqrt{2})$
   (b) $\mathbb{Q}(i) \subset \mathbb{Q}(i, \sqrt[3]{2})$
   (c) $\mathbb{Q} \subset \mathbb{Q}(i, \sqrt[3]{2})$

2. Find the minimal polynomial of the following simple extensions of $\mathbb{Q}$:
   (a) $\mathbb{Q}(i\sqrt[3]{5})$
   (b) $\mathbb{Q}(i + \sqrt{2})$
   (c) $\mathbb{Q}(\sqrt{3} + \sqrt{5})$

3. Let $\omega = -\frac{1+\sqrt{3}}{2}$, a cube root of 1. Prove that the fields $\mathbb{Q}(\sqrt[3]{2})$, $\mathbb{Q}(\omega\sqrt[3]{2})$, and $\mathbb{Q}(\omega^2\sqrt[3]{2})$ are all isomorphic to each other, but are all distinct.

4. (a) We have seen before that for any ring R, the set $\mathrm{Hom}(\mathbb{Z}, R)$ consists of one element. Prove that if R is a field, the kernel of this unique map is a prime ideal. By definition, the **characteristic** of a field is the unique nonnegative integer which generates this kernel.
   (b) Prove that $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ all have characteristic zero, and $\mathbb{F}_p$ has characteristic p.

5. This problem investigates finite fields.
   (a) Write down a list of all polynomials of degree 3 or less in $\mathbb{F}_2[x]$.
   (b) By plugging in values, or looking for multiples of smaller polynomials, circle those which are irreducible.
   (c) For each irreducible f in part (b), the quotient $\mathbb{F}_2[x]/(f)$ is a field (why?). Say, for each f in your list, how many elements this field has.
   (d) Make a conjecture on the possible sizes[1] of simple extensions of $\mathbb{F}_2$.
   (e) * Prove your conjecture, using the assumption that for every integer $r > 0$, there exists an irreducible polynomial of degree r over $\mathbb{F}_2$.
   (f) * Prove the assumption used in (e) that there exist irreducibles of all degrees in $\mathbb{F}_2[x]$.

[1] Here size means number of elements, not degree.