Math 110, Summer 2012: **Practice Exam 2**

Choose 3/4 of the following problems. Make sure to justify all steps in your solutions.

1. Let $A \in Mat_n(\mathbb{C})$.

i) Define the representation of $\mathbb{C}[t]$ determined by A, ρ_A. Define the minimal polynomial μ_A of A.

ii) What is the statement of the division algorithm for $\mathbb{C}[t]$?

iii) Let $f \in \ker \rho_A$ be nonzero. Prove that μ_A divides f.

For iv) - vi) let

$$A = \begin{bmatrix} -2 & 0 & 1 & -1 \\ 0 & -2 & -1 & 1 \\ 1 & 1 & 1 & 3 \\ -1 & -1 & 3 & 1 \end{bmatrix}.$$

iv) Show that

$$\mu_A = (t+2)^3(t-4).$$

v) Let $U_1 = \ker T_{(A+2I_4)^3}$, $U_2 = \ker T_{A-4I_4}$. Determine a basis $\mathcal{B} \subset U_1$ and the matrix $N = [f]_{\mathcal{B}}$, where

$$f : U_1 \to U_1 \ ; \ u \mapsto Au + 2u.$$

vi) f is nilpotent (you DO NOT have to show this). Determine a basis $\mathcal{C} \subset U_1$ such that $[f]_{\mathcal{C}}$ is block diagonal, each block being a 0-Jordan block.

vii) Determine a matrix $P \in GL_4(\mathbb{C})$ such that $P^{-1}AP$ is in Jordan canonical form.

Solution:

i) We define the representation of $\mathbb{C}[t]$ determined by A to be

$$\rho_A : \mathbb{C}[t] \to Mat_n(\mathbb{C}) \ ; \ f = a_0 + a_1 t + ... + a_k t^k \mapsto a_0 I_n + a_1 A + ... + a_k A^k.$$

The minimal polynomial $\mu_A \in \mathbb{C}[t]$ is the unique nonzero polynomial in $\ker \rho_A$ of minimal degree and leading coefficient 1.

ii) Let $f, g \in \mathbb{C}[t]$, $\deg f \leq \deg g$. Then, there exists $h \in \mathbb{C}[t]$ and $r \in \mathbb{C}[t]$, with $\deg r < \deg f$ such that

$$g = hf + r.$$

iii) Let $f \in \ker \rho_A$ be nonzero. Then, since $\deg \mu_A$ is of minimal degree we must have $\deg \mu_A \leq \deg f$. Using the division algorithm we can find $h \in \mathbb{C}[t]$, $r \in \mathbb{C}[t]$ where $\deg r < \deg \mu_A$, such that

$$f = \mu_A h + r.$$

Then, recalling that $\mu_A \in \ker \rho_A$, we obtain

$$0_n = \rho_A(f) = \rho_A(\mu_A h + r) = \rho_A(\mu_A)\rho_A(h) + \rho_A(r) = 0_n + \rho_A(r).$$

Hence, we have $r \in \ker \rho_A$. If r is nonzero then we would contradict the minimal degree property of $\mu_A \in \ker \rho_A$, so that $r = 0 \in \mathbb{C}[t]$ and $f = \mu_A h$.

iv) You can check that

$$\chi_A(t) = (t+2)^3(t-4),$$

and

$$(A + 2I_4)^2(A - 4I_4) \neq 0_4,$$

so that we must have

$$\mu_A = \chi_A.$$

v) We have

$$(A + 2I_4)^3 \sim \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so that

$$\ker T_{(A+2I_4)^3} = \text{span}_{\mathbb{C}}\{e_1, e_2, e_3 - e_4\}.$$

Hence, $\mathcal{B} = (e_1, e_2, e_3 - e_4) \subset U_1$ is a basis of U_1. Then,

$$N = [f]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix}.$$

vi) You can check that $N^3 = 0$, so that the exponent of N is $\eta(N) = 3$. Hence, the partition associated to N is $\pi(N) : 3$ (since the exponent of N is the largest integer appearing in the partition $\pi(N)$). Then, we note that

$$N^2 = \begin{bmatrix} 2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and proceed with the algorithm from section 2.3 of the notes: we have

$$H_3 = \mathbb{C}^3, \; H_2 = \text{span}_{\mathbb{C}}\{e_1 - e_2, e_3\},$$

and we can take

$$H_3 = H_2 \oplus G_3 = H_2 \oplus \text{span}_{\mathbb{C}}\{e_1\}.$$

Hence, we obtain the table

$$\begin{array}{c} e_1 \\ Ne_1 = e_3 \\ N^2e_1 = 2e_1 - 2e_2 \end{array},$$

so that we take the basis

$$\mathcal{C} = (2e_1 - 2e_2, e_3 - e_4, e_1) \subset U_1.$$

Then,

$$[f]_{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

vii) If we consider the basis

$$\mathcal{A} = \mathcal{C} \cup \mathcal{C}',$$

where $\mathcal{C}' = (e_3 + e_4) \subset U_2$, then we take the matrix

$$P = \begin{bmatrix} 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad (= P_{\mathcal{S}^{(4)} \leftarrow \mathcal{A}}$$

and see that

$$P^{-1}AP = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

2. i) Let V be a finite dimensional $\mathbb{K}$-vector space, $\mathbb{K}$ a number field. Define what it means for a function

$$B : V \times V \to \mathbb{K},$$

to be a $\mathbb{K}$-bilinear form on V.

ii) Define what it means for a $\mathbb{K}$-bilinear form B to be nondegenerate.

iii) Let $\mathcal{B} = (b_1, \dots, b_n) \subset V$ be an ordered basis of V, B a $\mathbb{K}$-bilinear form on V. Define the matrix of B with respect to $\mathcal{B}$. What is the fundamental relation between $B(u, v)$ and $[B]_{\mathcal{B}}$, for any $u, v \in V$?

iv) Let B be a $\mathbb{K}$-bilinear form on V, $\mathcal{B} \subset V$ an ordered basis of V. Prove that if $[B]_{\mathcal{B}}$ is invertible then B is nondegenerate.

v) Consider the bilinear form

$$B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \; ; \; (\underline{u}, \underline{v}) \mapsto \det([\underline{u} \; \underline{v}]), \text{ where } [\underline{u} \; \underline{v}] \text{ is the matrix with columns } \underline{u}, \underline{v}.$$

Is B nondegenerate? Justify your answer.

Solution:

i) B is a bilinear form if

- for every $u, v, w \in V$, $\lambda \in \mathbb{K}$, we have $B(u + \lambda v, w) = B(u, w) + \lambda B(v, w)$,
- for every $u, v, w \in V, \lambda \in \mathbb{K}$, we have $B(u, v + \lambda w) = B(u, v) + \lambda B(u, w)$.

ii) B is nondegenerate if,

$$B(u, v) = 0, \text{ for every } u \in V \implies v = 0_V.$$

iii) We define

$$[B]_{\mathcal{B}} = [a_{ij}], \text{ where } a_{ij} = B(b_i, b_j).$$

For any $u, v \in V$ we have

$$B(u, v) = [u]_{\mathcal{B}}^t [B]_{\mathcal{B}} [v]_{\mathcal{B}}.$$

iv) Suppose that $[B]_{\mathcal{B}}$ is invertible. Assume that $v \in V$ is such that

$$B(u, v) = 0, \text{ for every } u \in V.$$

Then, we must have, for every $u \in V$,

$$0 = [u]_{\mathcal{B}}^t [B]_{\mathcal{B}} [v]_{\mathcal{B}} \implies 0 = e_i^t [B]_{\mathcal{B}} [v]_{\mathcal{B}}, \text{ for each } i = 1, \dots, n.$$

Hence, if we denote

$$[B]_{\mathcal{B}}[v]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

then the previous relation implies that

$$x_i = 0, \text{ for every } i.$$

Hence, $[B]_{\mathcal{B}}[v]_{\mathcal{B}} = \underline{0}$ so that $[v]_{\mathcal{B}} = \underline{0}$, as $[B]_{\mathcal{B}}$ is invertible, and $v = 0_V$ (since the $\mathcal{B}$-coordinate morphism is injective).

v) Let $\mathcal{S}^{(2)} = (e_1, e_2)$ be the standard ordered basis of $\mathbb{R}^2$. Then, we have

$$B(e_1, e_1) = \det \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0, \ B(e_1, e_2) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1, \ B(e_2, e_1) = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1, \ B(e_2, e_2) = \det \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0,$$

so that

$$[B]_{\mathcal{S}^{(2)}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since this matrix is invertible then we must have that B is nondegenerate.

3. i) Let V be a finite dimensional $\mathbb{R}$-vector space, B a symmetric $\mathbb{R}$-bilinear form on V. Define what it means for B to be an inner product.

ii) Consider the bilinear form

$$B : Mat_2(\mathbb{R}) \times Mat_2(\mathbb{R}) \to \mathbb{R} \ ; \ (A, B) \mapsto \text{tr}(A^t X B), \text{ where } X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Without using the matrix of B with respect to some basis, show that B is symmetric. (*Hint: You may find the following facts useful:* $\text{tr}(A) = \text{tr}(A^t)$, *for* $A \in Mat_2(\mathbb{R})$, $\text{tr}(UV) = \text{tr}(VU)$, *for* $U, V \in Mat_2(\mathbb{R})$.)

iii) Determine the matrix of B, $[B]_{\mathcal{S}}$, with respect to the standard basis $\mathcal{S} = (e_{11}, e_{12}, e_{21}, e_{22})$,

iv) Determine the canonical form of B: ie, determine $P \in \text{GL}_4(\mathbb{R})$ such that

$$P^t[B]_{\mathcal{S}}P = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}, \ d_i \in \{1, -1\}.$$

v) Find $C \in Mat_2(\mathbb{R})$ such that $B(C, C) < 0$. Explain why B is not an inner product.

Solution:

i) B is an inner product if

$$B(v, v) \geq 0, \text{ for every } v \in V, \text{ and } B(v, v) = 0 \Leftrightarrow v = 0_V.$$

ii) Let $A, B \in Mat_2(\mathbb{R})$ then

$$B(A, B) = \text{tr}(A^t X B) = \text{tr}((A^t X B)^t) = \text{tr}(B^t X A) = B(B, A),$$

where we have used that $X^t = X$. Hence, B is symmetric.

iii) We have

$$[B]_{\mathcal{S}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

iv) We have

$$\underline{x}^t[B]_{\mathcal{S}}\underline{x} = x_1^2 + 2x_1x_3 + x_2^2 + 2x_2x_4 = (x_1+x_3)^2 + (x_2+x_4)^2 - x_3^2 - x_4^2.$$

Set

$$y_1 = x_1 + x_3,\ y_2 = x_2 + x_4,\ y_3 = x_3,\ y_4 = x_4,$$

so that we have

$$\underline{y} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\underline{x}. \ (= Q\underline{x})$$

Let

$$P = Q^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, we have

$$P^t[B]_{\mathcal{S}}P = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}.$$

v) Set

$$C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

Then, we have

$$B(C, C) = \mathrm{tr} C^t X C = -1.$$

Hence, B is not an inner product as the condition defined in i) is not satisfied.

4. i) Let $(V, \langle , \rangle)$ be a Euclidean space. Define the notion of the length of vector $v \in V$.

ii) Define what it means for a linear morphism $f : V \to V$ to be

a) a Euclidean morphism,

b) an orthogonal transformation.

Prove that if $f : V \to V$ is a Euclidean morphism then f is an orthogonal transformation.

iii) Prove that if $f \in O(\mathbb{E}^n)$ is an orthogonal transformation, $\mathcal{B} \subset \mathbb{R}^n$ is an ordered basis of $\mathbb{R}^n$, then $A = [f]_{\mathcal{B}}$ satisfies $A^tA = I_n$.

iv) Let $S \subset \mathbb{E}^4$ be a nonempty subset. Define what it means for S to orthogonal.

v) Determine an orthogonal basis of $\ker f$, where

$$f : \mathbb{R}^4 \to \mathbb{R}^4 \ ; \ \underline{x} \mapsto \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}\underline{x}.$$

Here we are assuming that orthogonality is with respect to the 'dot prduct' on $\mathbb{R}^4$.

Explain why f is not a Euclidean morphism.

vi) Determine the orthogonal complement W of $\ker f$.

Solution:

i) We define $||v|| = \sqrt{\langle v, v\rangle}$.

ii) a) for every $u, v \in V$ we have $\langle u, v\rangle = \langle f(u), f(v)\rangle$,

b) f is an isomorphism and a Euclidean morphism.

iii) ???? This question is not well-posed, sorry!

iv) S is orthogonal if, for every $s, t \in S, s \neq t$, we have

$$s \cdot t = 0.$$

v) We have

$$\ker f = \text{span}_{\mathbb{R}} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Using Gram-Schmidt we obtain an orthogonal basis

$$\begin{aligned} c_1 &= \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \\ c_2 &= \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

f is not a Euclidean morphism since f is not injective: all Euclidean morphisms are injective.

vi) $\underline{x} \in W^{\perp}$ if and only if

$$\underline{x} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 0 = \underline{x} \cdot \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Hence, we must have

$$-x_1 - x_2 + x_3 = 0, \ -x_1 - x_3 + x_4 = 0.$$

As we have

$$\begin{bmatrix} -1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix},$$

then we have

$$W^{\perp} = \text{span}_{\mathbb{R}} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$