Math 110, Summer 2012: **Practice Exam 2**

Choose 3/4 of the following problems. Make sure to justify all steps in your solutions.

1. Let $A \in Mat_n(\mathbb{C})$.

i) Define the representation of $\mathbb{C}[t]$ determined by A, ρ_A. Define the minimal polynomial μ_A of A.

ii) What is the statement of the division algorithm for $\mathbb{C}[t]$?

iii) Let $f \in \ker \rho_A$ be nonzero. Prove that μ_A divides f.

For iv) - vi) let

$$A = \begin{bmatrix} -2 & 0 & 1 & -1 \\ 0 & -2 & -1 & 1 \\ 1 & 1 & 1 & 3 \\ -1 & -1 & 3 & 1 \end{bmatrix}.$$

iv) Show that

$$\mu_A = (t+2)^3(t-4).$$

v) Let $U_1 = \ker T_{(A+2I_4)^3}$, $U_2 = \ker T_{A-4I_4}$. Determine a basis $\mathcal{B} \subset U_1$ and the matrix $N = [f]_{\mathcal{B}}$, where

$$f : U_1 \to U_1 \; ; \; u \mapsto Au + 2u.$$

vi) f is nilpotent (you DO NOT have to show this). Determine a basis $\mathcal{C} \subset U_1$ such that $[f]_{\mathcal{C}}$ is block diagonal, each block being a 0-Jordan block.

vii) Determine a matrix $P \in GL_4(\mathbb{C})$ such that $P^{-1}AP$ is in Jordan canonical form.

2. i) Let V be a finite dimensional $\mathbb{K}$-vector space, $\mathbb{K}$ a number field. Define what it means for a function

$$B : V \times V \to \mathbb{K},$$

to be a $\mathbb{K}$-bilinear form on V.

ii) Define what it means for a $\mathbb{K}$-bilinear form B to be nondegenerate.

iii) Let $\mathcal{B} = (b_1, \dots, b_n) \subset V$ be an ordered basis of V, B a $\mathbb{K}$-bilinear form on V. Define the matrix of B with respect to $\mathcal{B}$. What is the fundamental relation between $B(u, v)$ and $[B]_{\mathcal{B}}$, for any $u, v \in V$?

iv) Let B be a $\mathbb{K}$-bilinear form on V, $\mathcal{B} \subset V$ an ordered basis of V. Prove that if $[B]_{\mathcal{B}}$ is invertible then B is nondegenerate.

v) Consider the bilinear form

$$B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \; ; \; (\underline{u}, \underline{v}) \mapsto \det([\underline{u} \; \underline{v}]), \text{ where } [\underline{u} \; \underline{v}] \text{ is the matrix with columns } \underline{u}, \underline{v}.$$

Is B nondegenerate? Justify your answer.

3. i) Let V be a finite dimensional $\mathbb{R}$-vector space, B a symmetric $\mathbb{R}$-bilinear form on V. Define what it means for B to be an inner product.

ii) Consider the bilinear form

$$B : Mat_2(\mathbb{R}) \times Mat_2(\mathbb{R}) \to \mathbb{R} \; ; \; (A, B) \mapsto \mathrm{tr}(A^t XB), \text{ where } X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Without using the matrix of B with respect to some basis, show that B is symmetric. (*Hint: You may find the following facts useful:* $\mathrm{tr}(A) = \mathrm{tr}(A^t)$, *for* $A \in Mat_2(\mathbb{R})$, $\mathrm{tr}(UV) = \mathrm{tr}(VU)$, *for* $U, V \in Mat_2(\mathbb{R})$.)

iii) Determine the matrix of B, $[B]_{\mathcal{S}}$, with respect to the standard basis $\mathcal{S} = (e_{11}, e_{12}, e_{21}, e_{22})$,

iv) Determine the canonical form of B: ie, determine $P \in \mathrm{GL}_4(\mathbb{R})$ such that

$$P^t [B]_{\mathcal{S}} P = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}, \; d_i \in \{1, -1\}.$$

v) Find $C \in Mat_2(\mathbb{R})$ such that $B(C, C) < 0$. Explain why B is not an inner product.

4. i) Let $(V, \langle , \rangle)$ be a Euclidean space. Define the notion of the length of vector $v \in V$.

ii) Define what it means for a linear morphism $f : V \to V$ to be

a) a Euclidean morphism,

b) an orthogonal transformation.

Prove that if $f : V \to V$ is a Euclidean morphism then f is an orthogonal transformation.

iii) Prove that if $f \in O(\mathbb{E}^n)$ is an orthogonal transformation, $\mathcal{B} \subset \mathbb{R}^n$ is an ordered basis of $\mathbb{R}^n$, then $A = [f]_{\mathcal{B}}$ satisfies $A^t A = I_n$.

iv) Let $S \subset \mathbb{E}^4$ be a nonempty subset. Define what it means for S to orthogonal.

v) Determine an orthogonal basis of $\ker f$, where

$$f : \mathbb{R}^4 \to \mathbb{R}^4 \; ; \; \underline{x} \mapsto \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix} \underline{x}.$$

Here we are assuming that orthogonality is with respect to the 'dot prduct' on $\mathbb{R}^4$.

Explain why f is not a Euclidean morphism.

vi) Determine the orthogonal complement W of $\ker f$.