Math 110, Summer 2012: **Practice Exam 1**

Choose 3/5 of the following problems. Make sure to justify all steps in your solutions.

1. Let V be a $\mathbb{K}$-vector space, for some number field $\mathbb{K}$. Let $U \subset V$ be a nonempty subset of V.

i) Define what it means for $U \subset V$ to be a vector subspace of V. Define $\text{span}_{\mathbb{K}} U$.

ii) Prove that $\text{span}_{\mathbb{K}} U$ is a vector subspace of V.

iii) Consider the $\mathbb{Q}$-vector space $V = Mat_2(\mathbb{Q})$ and the subset

$$U = \{I_2, A, A^2\}, \quad \text{where} \quad A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Find $v \in U$ such that

$$\text{span}_{\mathbb{Q}} U = \text{span}_{\mathbb{Q}} U',$$

where $U' = U \setminus \{v\}$. Show that U' is linearly independent.

iv) Find a vector $w \in Mat_2(\mathbb{Q})$ such that $w \notin \text{span}_{\mathbb{Q}} U$. Is the set $U' \cup \{w\}$ linearly independent? Explain your answer.

v) Extend the set $U' \cup \{w\}$ to a basis $\mathcal{B}$ of $Mat_2(\mathbb{Q})$, taking care to explain how you know the set $\mathcal{B}$ you've obtained is a basis.

2. i) Let $\mathcal{B} = (b_1, \dots, b_n) \subset V$ be an ordered subset of the $\mathbb{K}$-vector space V. Define what it means for $\mathcal{B}$ to be an ordered basis of V using the notions of linear independence AND span.

ii) Consider the maximal linear independence property: *let $E \subset V$ be a linearly independent subset of the $\mathbb{K}$-vector space V. Then, if $E \subset E'$ and E' is linearly independent then $E' = E$.*

Prove that if $\mathcal{B} \subset V$ is a basis (satisfying the definition you gave in 2a)) then $\mathcal{B}$ is maximal linearly independent.

iii) Consider the $\mathbb{Q}$-vector space $\mathbb{Q}^{\{1,2,3\}} = \{f : \{1, 2, 3\} \to \mathbb{Q}\}$. Let

$$\mathcal{B} = \{f_1, f_2, f_3\} \subset \mathbb{Q}^{\{1,2,3\}},$$

where

$$f_1(1) = 0, f_1(2) = -1, f_1(3) = 1, \ f_2(1) = 0, f_2(2) = 1, f_2(3) = 1, \ f_3(1) = 1, f_3(2) = 1, f_3(3) = 1.$$

Prove that $\mathcal{B}$ is a basis of $\mathbb{Q}^{\{1,2,3\}}$.

iv) Let $\mathcal{B}$ be as defined in 2iii). Define the $\mathcal{B}$-coordinate morphism

$$[-]_{\mathcal{B}} : \mathbb{Q}^{\{1,2,3\}} \to \mathbb{Q}^3,$$

and determine the $\mathcal{B}$-coordinates of $f \in \mathbb{Q}^{\{1,2,3\}}$, where

$$f(1) = 2, f(2) = -1, f(3) = 2.$$

v) Suppose that $\mathcal{C} = (c_1, c_2, c_3) \subset \mathbb{Q}^{\{1,2,3\}}$ is an ordered basis such that the change of coordinate matrix

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Determine $c_1, c_2, c_3 \in \mathbb{Q}^{\{1,2,3\}}$, ie, for each $i \in \{1, 2, 3\}$, determine $c_1(i), c_2(i), c_3(i)$.

3. i) Define the kernel $\ker f$ of a linear morphism $f : V \to W$ and the rank of f, $\mathrm{rank} f$.

ii) Consider the function

$$f : \mathbb{Q}^3 \to \mathbb{Q}^2 \; ; \; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - x_2 + 2x_3 \\ x_1 + x_3 \end{bmatrix}.$$

Explain briefly why f is a linear morphism. What is the rank of f? Justify your answer. Using only the rank of f prove that f is not injective (do not row-reduce!).

iii) Let $\mathcal{S}^{(2)} \subset \mathbb{Q}^2, \mathcal{S}^{(3)} \subset \mathbb{Q}^3$ be the standard ordered bases. Find invertible matrices $P \in \mathrm{GL}_2(\mathbb{Q})$, $Q \in \mathrm{GL}_3(\mathbb{Q})$ such that

$$Q^{-1}[f]_{\mathcal{S}^{(3)}}^{\mathcal{S}^{(2)}} P = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where $r = \mathrm{rank} f$.

iv) Prove or disprove: for the P you obtained in 3iii), a column of P is a basis of $\ker f$.

4. i) Let $A \in Mat_n(\mathbb{C})$. Define what it means for A to be diagonalisable.

ii) Suppose that $P^{-1}AP = D$, with D a diagonal matrix and $P \in \mathrm{GL}_n(\mathbb{C})$. Prove that the columns of P are eigenvectors of A.

For 4iii)-vi) we assume that $A \in Mat_2(\mathbb{C})$, $A^2 = I_2$ and that A is NOT a diagonal matrix.

iii) Show that the only possible eigenvalues of A are $\lambda = 1$ or $\lambda = -1$.

iv) Let $u \in \mathbb{C}^2$ be nonzero. Show that $A(Au + u) = Au + u$.

v) Prove that there always exists some nonzero $w \in \mathbb{C}^2$ such that $Aw \neq -w$. Deduce that $\lambda = 1$ must occur as an eigenvalue of A. Prove that $\lambda = 1$ must also occur as an eigenvalue of A.

vi) Deduce that A is diagonalisable.

5. i) Define what it means for a linear endomorphism $f \in \mathrm{End}_{\mathbb{C}}(V)$ to be nilpotent. Define the exponent of f, $\eta(f)$.

ii) Consider the endomorphism

$$f : \mathbb{C}^3 \to \mathbb{C}^3 \; ; \; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 + x_2 - x_3 \\ 0 \end{bmatrix}.$$

Show that f is nilpotent and determine the exponent of f, $\eta(f)$.

iii) Define the height of a vector $v \in \mathbb{C}^3$ (with respect to f), $\mathrm{ht}(v)$. Find a vector $v \in \mathbb{C}^3$ such that $\mathrm{ht}(v) = 2$.

iv) Find a determine a basis $\mathcal{B}$ of $\mathbb{C}^3$ such that

$$[f]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

v) What is the partition of 3 corresponding to the similarity class of f?