Then, the adjoint of f is the morphism

$$f^{+} : \mathbb{Q}^3 \to \mathbb{Q}^3 \ ; \ \underline{x} \mapsto \begin{bmatrix} 1 & -3 & -1 \\ 1 & 5 & 0 \\ 0 & 2 & 3 \end{bmatrix} \underline{x}.$$

As a verification, you can check that

$$B\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix} 1 & -3 & -1 \\ 1 & 5 & 0 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix}-1\\0\\-1\end{bmatrix}\right) = B\left(\begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 0 \\ -3 & 2 & 5 \end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\end{bmatrix}\right).$$

3.2 Real and complex symmetric bilinear forms

Throughout the remainder of these notes we will assume that $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Throughout this section we will assume that all bilinear forms are symmetric.

When we consider symmetric bilinear forms on real or complex vector spaces we obtain some particularly nice results.[60] For a $\mathbb{C}$-vector space V and symmetric bilinear form $B \in \mathrm{Bil}_{\mathbb{C}}(V)$ we will see that there is a basis $\mathcal{B} \subset V$ such that

$$[B]_{\mathcal{B}} = I_{\dim V}.$$

First we introduce the important *polarisation identity.*

Lemma 3.2.1 (Polarisation identity). *Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ be a symmetric bilinear form. Then, for any $u, v \in V$, we have*

$$B(u, v) = \frac{1}{2}\left(B(u + v, u + v) - B(u, u) - B(v, v)\right).$$

Proof: Left as an exercise for the reader. □

Corollary 3.2.2. *Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ be symmetric and nonzero. Then, there exists some nonzero $v \in V$ such that $B(v, v) \neq 0$.*

Proof: Suppose that the result does not hold: that is, for every $v \in V$ we have $B(v, v) = 0$. Then, using the polarisation identity (Lemma 3.2.1) we have, for every $u, v \in V$,

$$B(u, v) = \frac{1}{2}\left(B(u + v, u + v,) - B(u, u) - B(v, v)\right) = \frac{1}{4}(0 - 0 - 0) = 0.$$

Hence, we must have that $B = 0$ is the zero bilinear form, which contradicts our assumption on B. Hence, ther must exist some $v \in V$ such that $B(v, v) \neq 0$. □

This seemingly simple result has some profound consequences for nondegenerate complex symmetric bilinear forms.

Theorem 3.2.3 (Classification of nondegenerate symmetric bilinear forms over $\mathbb{C}$). *Let $B \in \mathrm{Bil}_{\mathbb{C}}(V)$ be symmetric and nondegenerate. Then, there exists an ordered basis $\mathcal{B} \subset V$ such that*

$$[B]_{\mathcal{B}} = I_{\dim V}.$$

Proof: By Corollary 3.2.2 we know that there exists some nonzero $v_1 \in V$ such that $B(v_1, v_1) \neq 0$ (we know that B is nonzero since it is nondegenerate). Let $E_1 = \mathrm{span}_{\mathbb{C}}\{v_1\}$ and consider $E_1^{\perp} \subset V$.

We have $E_1 \cap E_1^{\perp} = \{0_V\}$: indeed, let $x \in E_1 \cap E_1^{\perp}$. Then, $x = cv_1$, for some $c \in \mathbb{C}$. As $x \in E_1^{\perp}$ we must have

$$0 = B(x, v_1) = B(cv_1, v_1) = cB(v_1, v_1),$$

so that $c = 0$ (as $B(v_1, v_1) \neq 0$). Thus, by Proposition 3.1.17, we must have

$$V = E_1 \oplus E_1^{\perp}.$$

[60] Actually, all results that hold for $\mathbb{C}$-vector space also hold for $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is an algebraically closed field. To say that $\mathbb{K}$ is algebraically closed means that the Fundamental Theorem of Algebra holds for $\mathbb{K}[t]$; equivalently, every polynomial $f \in \mathbb{K}[t]$ can be written as a product of linear factors.

Moreover, B restricts to a nondegenerate symmetric bilinear form on $E_1^\perp$: indeed, the restriction is

$$B_{|E_1^\perp} : E_1^\perp \times E_1^\perp \to \mathbb{C} \; ; \; (u, u') \mapsto B(u, u'),$$

and this is a symmetric bilinear form. We need to check that it is nondegenerate. Suppose that $w \in E_1^\perp$ is such that, for every $z \in E_1^\perp$ we have

$$B(z, w) = 0.$$

Then, for any $v \in V$, we have $v = cv_1 + z, z \in E_1^\perp, c \in \mathbb{C}$, so that

$$B(v, w) = B(cv_1 + z, w) = cB(v_1, w) + B(z, w) = 0 + 0 = 0,$$

where we have used the assumption on w and that $w \in E_1^\perp$. Hence, using nongeneracy of B on V we see that $w = 0_V$. Hence, we have that B is also nondegenerate on $E_1^\perp$.

As above, we can now find $v_2 \in E_1^\perp$ such that $B(v_2, v_2) \neq 0$ and, if we denote $E_2 = \mathrm{span}_{\mathbb{C}}\{v_2\}$, then

$$E_1^\perp = E_2 \oplus E_2^\perp,$$

where $E_2^\perp$ is the B-complement of E_2 **in** $E_1^\perp$. Hence, we have

$$V = E_1 \oplus E_2 \oplus E_2^\perp.$$

Proceeding in the manner we obtain

$$V = E_1 \oplus \cdots \oplus E_n,$$

where $n = \dim V$, and where $E_i = \mathrm{span}_{\mathbb{C}}\{v_i\}$. Moreover, by construction we have that

$$B(v_i, v_j) = 0, \quad \text{for } i \neq j.$$

Define

$$b_i = \frac{1}{\sqrt{B(v_i, v_i)}} v_i;$$

we know that the square root $\sqrt{B(v_i, v_i)}$ exists (and is nonzero) since we are considering $\mathbb{C}$-scalars.[61] Then, it is easy to see that

$$B(b_i, b_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Finally, since

$$V = \mathrm{span}_{\mathbb{C}}\{b_1\} \oplus \cdots \oplus \mathrm{span}_{\mathbb{C}}\{b_n\},$$

we have that $\mathcal{B} = (b_1, \dots, b_n)$ is an ordered basis such that

$$[B]_{\mathcal{B}} = I_n.$$

□

Corollary 3.2.4. *Let $A \in \mathrm{GL}_n(\mathbb{C})$ be a symmetric matrix (so that $A = A^t$). Then, there exists $P \in \mathrm{GL}_n(\mathbb{C})$ such that*

$$P^t A P = I_n.$$

Proof: This is just Theorem3.2.3 and Proposition 3.1.8 applied to the bilinear form $B_A \in \mathrm{Bil}_{\mathbb{C}}(\mathbb{C}^n)$. The assumptions on A ensure that B_A is symmetric and nondegenerate. □

Corollary 3.2.5. *Suppose that $X, Y \in \mathrm{GL}_n(\mathbb{C})$ are both symmetric. Then, there is a nondegenerate bilinear form $B \in \mathrm{Bil}_{\mathbb{C}}(\mathbb{C}^n)$ and bases $\mathcal{B}, \mathcal{C} \subset \mathbb{C}^n$ such that*

$$X = [B]_{\mathcal{B}}, \; Y = [B]_{\mathcal{C}}.$$

[61] This is a consequence of the Fundamental Theorem of Algebra: for any $c \in \mathbb{C}$ we have that

$$t^2 - c = 0,$$

has a solution.

Proof: By the previous Corollary we can find $P, Q \in \mathrm{GL}_n(\mathbb{C})$ such that

$$P^t X P = I_n = Q^t Y Q \implies (Q^{-1})^t P^t X P Q^{-1} = Y \implies (PQ^{-1})^t X P Q^{-1} = Y.$$

Now, let $B = B_X \in \mathrm{Bil}_{\mathbb{C}}(\mathbb{C}^n)$, $\mathcal{B} = \mathcal{S}^{(n)}$ and $\mathcal{C} = (c_1, \dots, c_n)$, where c_i is the i^{th} column of PQ^{-1}. Then, the above identity states that

$$[B]_{\mathcal{C}} = P^t_{\mathcal{B}\leftarrow\mathcal{C}}[B]_{\mathcal{B}} P_{\mathcal{B}\leftarrow\mathcal{C}} = Y.$$

The result follows. □

The situation is not as simple for an $\mathbb{R}$-vector space V and nondegenerate symmetric bilinear form $B \in \mathrm{Bil}_{\mathbb{R}}(V)$, however we can still obtain a nice classification result.

Theorem 3.2.6 (Sylvester's law of inertia)**.** *Let V be an $\mathbb{R}$-vector space, $B \in \mathrm{Bil}_{\mathbb{R}}(V)$ a nondegenerate symmetric bilinear form. Then, there is an ordered basis $\mathcal{B} \subset V$ such that $[B]_{\mathcal{B}}$ is a diagonal matrix*

$$[B]_{\mathcal{B}} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix},$$

where $d_i \in \{1, -1\}$.

Moreover, if $p =$ the number of 1s appearing on the diagonal and $q =$ the number of -1s appearing on the diagonal, then p and q are invariants of B: this means that if $\mathcal{C} \subset V$ is any other basis of V such that

$$[B]_{\mathcal{C}} = \begin{bmatrix} e_1 & & & \\ & e_2 & & \\ & & \ddots & \\ & & & e_n \end{bmatrix},$$

where $e_j \in \{1, -1\}$, and p' (resp. q') denotes the number of 1s (resp. -1s) on the diagonal. Then,

$$p = p', \; q = q'.$$

Proof: The proof is similar to the proof of Theorem 3.2.3: we determine $v_1, \dots, v_n \in V$ such that

$$V = \mathrm{span}_{\mathbb{R}}\{v_1\} \oplus \cdots \oplus \mathrm{span}_{\mathbb{R}}\{v_n\},$$

and with $B(v_i, v_j) = 0$, whenever $i \neq j$. However, we now run into a problem: what if $B(v_i, v_i) < 0$? We can't find a real square root of a negative number so we can't proceed as in the complex case. However, if we define

$$\delta_i = \sqrt{|B(v_i, v_i)|}, \quad \text{for every } i,$$

then we can obtain a basis $\mathcal{B} = (b_1, \dots, b_n)$, where we define

$$b_i = \frac{1}{\delta_i} v_i.$$

Then, we see that

$$B(b_i, b_j) = \begin{cases} 0, & i \neq j, \\ \pm 1, & i = j, \end{cases}$$

and $[B]_{\mathcal{B}}$ is of the required form.

Let us reorder $\mathcal{B}$ so that, for $i = 1, \dots, p$, we have $B(b_i, b_i) > 0$. Then, if we denote

$$P = \mathrm{span}_{\mathbb{R}}\{b_1, \dots, b_p\}, \quad \text{and} \quad Q = \mathrm{span}_{\mathbb{R}}\{b_{p+1}, \dots, b_n\},$$

we have

$$\dim P = p, \; \dim Q = q \; (= n - p).$$

We see that the restriction of B to P satisfies

$$B(u,u) > 0, \text{ for every } u \in P,$$

and that if $P \subset P'$, $P \neq P'$, with $P' \subset V$ a subspace, then there is some $v \in P'$ such that $B(v,v) \leq 0$: indeed, as $v \notin P$ then we have

$$v = \lambda_1 b_1 + ... + \lambda_p b_p + \mu_1 b_{p+1} + ... + \mu_q b_n,$$

and some $\mu_j \neq 0$. Then, since $P \subset P'$ we must have $b_{p+j} \in P'$ and

$$B(b_{p+j}, b_{p+j}) < 0.$$

Hence, we can see that p is the dimension of the largest subspace U of V for which the restriction of B to U satisfies $B(u,u) > 0$, for every $u \in U$.

Similarly, we can define q to be the dimension of the largest subspace $U' \in V$ for which the restriction of B to U' satisfies $B(u',u') < 0$, for every $u' \in U'$.

Therefore, we have defined p and q only in terms of B so that they are invariants of B. □

Corollary 3.2.7. *For every symmetric $A \in \mathrm{GL}_n(\mathbb{R})$, there exists $X \in GL_n(\mathbb{R})$ such that*

$$X^t A X = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix},$$

with $d_i \in \{1, -1\}$.

Definition 3.2.8. Suppose that $B \in \mathrm{Bil}_{\mathbb{R}}(V)$ is nondegenerate and symmetric and that p, q are as in Theorem 3.2.6. Then, we define the *signature of B*, denoted $\mathrm{sig}(B)$, to be the number

$$\mathrm{sig}(B) = p - q.$$

It is an invariant of B: for any basis $\mathcal{B} \subset V$ such that

$$[B]_{\mathcal{B}} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix},$$

with $d_i \in \{1, -1\}$, the quantity $p - q$ is the same.

3.2.1 Computing the canonical form of a real nondegenerate symmetric bilinear form

([1], p.185-191)

Suppose that $B \in \mathrm{Bil}_{\mathbb{R}}(V)$ is **symmetric** and **nondegenerate**, with V a finite dimensional $\mathbb{R}$-vector space. Suppose that $\mathcal{B} \subset V$ is an ordered basis such that

$$[B]_{\mathcal{B}} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix},$$

where $d_i \in \{1, -1\}$. Such a basis exists by Theorem 3.2.6. How do we determine $\mathcal{B}$?

Suppose that $\mathcal{C} \subset V$ is **any** ordered basis. Then, we know that

$$P^t_{\mathcal{C} \leftarrow \mathcal{B}} [B]_{\mathcal{C}} P_{\mathcal{C} \leftarrow \mathcal{B}} = [B]_{\mathcal{B}},$$

by Proposition 3.1.8. Hence, the problem of determining $\mathcal{B}$ is equivalent to the problem of determining $P_{\mathcal{C}\leftarrow\mathcal{B}}$ (since we already know $\mathcal{C}$ and we can use $P_{\mathcal{C}\leftarrow\mathcal{B}}$ to determine $\mathcal{B}$[62]).

Therefore, suppose that $A = [a_{ij}] \in \mathrm{GL}_n(\mathbb{R})$ is symmetric. We want to determine $P \in \mathrm{GL}_n(\mathbb{R})$ such that

$$P^t A P = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix},$$

where $d_i \in \{1, -1\}$.

Consider the column vector of variables

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Then, we have

$$\underline{x}^t A \underline{x} = a_{11}x_1^2 + \dots + a_{nn}x_n^2 + 2\sum_{i<j} a_{ij}x_i x_j.^{63}$$

By performing the 'completing the square' process for each variable x_i we will find variables

$$\begin{aligned} y_1 &= \quad q_{11}x_1 + q_{12}x_2 + \dots + q_{1n}x_n, \\ y_2 &= \quad q_{21}x_1 + q_{22}x_1 + \dots + q_{2n}x_n \\ &\vdots \\ y_n &= \quad q_{n1}x_1 + q_{n2}x_2 + \dots + q_{nn}x_n \end{aligned}$$

such that

$$\underline{x}^t A \underline{x} = y_1^2 + \dots + y_p^2 - y_{p+1}^2 - \dots - y_n^2.$$

Then, $P = [q_{ij}]^{-1}$ is the matrix we are looking for.

Why? The above system of equations corresponds to the matrix equation

$$\underline{y} = Q\underline{x}, \quad Q = [q_{ij}] \in \mathrm{GL}_n(\mathbb{R}),$$

which we can consider as a change of coordiante transformation $P_{\mathcal{B}\leftarrow\mathcal{S}^{(n)}}$ from the standard basis $\mathcal{S}^{(n)} \subset \mathbb{R}^n$ to a basis $\mathcal{B}$ (we consider $\underline{x}$ to be the $\mathcal{S}^{(n)}$-coordinate vector of the corresponding element of $\mathbb{R}^n$). Then, we see that

$$(P\underline{y})^t A(P\underline{y}) = \underline{x}^t A \underline{x} = y_1^2 + \dots + y_p^2 - y_{p+1}^2 - \dots - y_n^2,$$

where $P = Q^{-1}$. As

$$\underline{y}^t P^t A P \underline{y} = (P\underline{y})^t A(P\underline{y}) = y_1^2 + \dots + y_p^2 - y_{p+1}^2 - \dots - y_n^2 = \underline{y}^t \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix} \underline{y},$$

we see that $P^t A P$ is of the desired form. Moreover, $\mathcal{B}$ is the required basis.

It is better to indicate this method through an example.

[62]Why?

[63]The assignment $\underline{x} \mapsto \underline{x}^t A \underline{x}$ is called a *quadratic form*. The study of quadratic forms and their properties is primarily determined by the symmetric bilinear forms defined by A.

Example 3.2.9. 1. Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & 1 & -2 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & -1 \end{bmatrix},$$

so that A is symmetric and invertible. Consider the column vector of variable $\underline{x}$ as above. Then, we have

$$\underline{x}^t A \underline{x} = x_1^2 + 2x_2^2 - x_4^2 - 2x_1x_3 + 4x_1x_4 + 2x_2x_3 - 4x_2x_4.$$

Let's complete the square with respect to x_1: we have

$$\begin{aligned} & x_1^2 + 2x_2^2 - x_4^2 - 2x_1x_3 + 4x_1x_4 + 2x_2x_3 - 4x_2x_4 \\ = {} & x_1^2 - 2x_1(x_3 - 2x_4) + (x_3 - 2x_4)^2 - (x_3 - 2x_4)^2 + 2x_2^2 - x_4^2 + 2x_2x_3 - 4x_2x_4 \\ = {} & (x_1 - (x_3 - 2x_4))^2 + 2x_2^2 - x_3^2 - 5x_4^2 + 2x_2x_3 - 4x_2x_4 + 4x_3x_4 \end{aligned}$$

Now we set

$$y_1 = x_1 - x_3 + 2x_4.$$

Then, complete the square with respect to the remaining x_2 terms: we have

$$\begin{aligned} & y_1^2 + 2x_2^2 - x_3^2 - 5x_4^2 + 2x_2x_3 - 4x_2x_4 + 4x_3x_4 \\ = {} & y_1^2 + 2(x_2^2 + x_2(x_3 - 2x_4) + \frac{1}{4}(x_3 - 2x_4)^2) - \frac{1}{2}(x_3 - 2x_4)^2 - x_3^2 - x_4^2 - 4x_3x_4 \\ = {} & y_1^2 + 2(x_2 + \frac{1}{2}(x_3 - 2x_4))^2 - \frac{3}{2}x_3^2 - 7x_4^2 - 2x_3x_4 \end{aligned}$$

Now we set

$$y_2 = \sqrt{2}\left(x_2 + \tfrac{1}{2}x_3 - x_4\right).$$

We obtain

$$x_1^2 + 2x_2^2 - x_4^2 - 2x_1x_3 + 4x_1x_4 + 2x_2x_3 - 4x_2x_4 = y_1^2 + y_2^2 - \frac{3}{2}x_3^2 - 7x_4^2 - 2x_3x_4.$$

Completing the square with respect to x_3 we obtain

$$\begin{aligned} & y_1^2 + y_2^2 - \frac{3}{2}x_3^2 - 7x_4^2 - 2x_3x_4 \\ = {} & y_1^2 + y_2^2 - \frac{3}{2}(x_3^2 + \frac{14}{3}x_3x_4 + \frac{49}{9}x_4^2) + \frac{49}{6}x_4^2 \\ = {} & y_1^2 + y_2^2 - \frac{3}{2}(x_3 + \frac{7}{3}x_4)^2 + \frac{49}{6}x_4^2. \end{aligned}$$

Then, set

$$\begin{aligned} y_3 &= \sqrt{\tfrac{3}{2}}(x_3 + \tfrac{7}{3}x_4), \\ y_4 &= \tfrac{7}{\sqrt{6}}x_4 \end{aligned}.$$

So, if we let

$$Q = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} & -\sqrt{2} \\ 0 & 0 & \sqrt{\frac{3}{2}} & \frac{7}{\sqrt{6}} \\ 0 & 0 & 0 & \frac{7}{\sqrt{6}} \end{bmatrix},$$

then we have

$$\underline{y} = Q\underline{x}.$$

Hence, if we define $P = Q^{-1}$, then we have that

$$P^t A P = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}.$$

Hence, we have that $p = 3, q = 1$ and that if $B_A \in \mathrm{Bil}_{\mathbb{R}}(\mathbb{R}^4)$ then

$$\mathrm{sig}(B_A) = 3 - 1 = 2.$$

2. Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

which is symmetric and invertible. Consider the column vector of variables $\underline{x}$ as before. Then, we have

$$\underline{x}^t A \underline{x} = -x_1^2 + 2x_2x_3.$$

Proceeding as before, we 'complete the square' with respect to x_2 (we don't need to complete the square for x_1): we have

$$\begin{aligned} & -x_1^2 + 2x_2x_3 \\ = & -x_1^2 + \frac{1}{2}(x_2 + x_3)^2 - \frac{1}{2}(x_2 - x_3)^2 \end{aligned}$$

Hence, if we let

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= \tfrac{1}{\sqrt{2}}(x_2 + x_3) \\ y_3 &= \tfrac{1}{\sqrt{2}}(x_2 - x_3) \end{aligned}$$

then we have

$$\underline{x}^t A \underline{x} = -y_1^2 + y_2^2 - y_3^2.$$

Furthermore, if we let

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix},$$

and defined $P = Q^{-1}$, then

$$P^t A P = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}.$$

Hence, $p = 1, q = 2$ and

$$\mathrm{sig}(B_A) = -1.$$

3.3 Euclidean spaces

Throughout this section V will be a finite dimensional $\mathbb{R}$-vector space and $\mathbb{K} = \mathbb{R}$.

Definition 3.3.1. Let $B \in \mathrm{Bil}_{\mathbb{R}}(V)$ be a symmetric bilinear form. We say that B is an *inner product on V* if B satisfies the following property:

$$B(v, v) \geq 0, \quad \text{for every } v \in V, \text{ and } \; B(v, v) = 0 \Leftrightarrow v = 0_V.$$

If $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ is an inner product on V then we will write

$$\langle u, v \rangle \stackrel{def}{=} B(u, v).$$