is a block diagonal matrix, with $A_i \in Mat_{\dim U_i}(\mathbb{C})$. In fact, we can assume that $\mathcal{B} = \mathcal{B}_1 \cup ... \cup \mathcal{B}_k$, with $\mathcal{B}_i$ an ordered basis of U_i, and that

$$A_i = [f_{|U_i}]_{\mathcal{B}_i},$$

where $f_{|U_i} : U_i \to U_i$ is the restriction of f to U_i.[40]

2.3 Nilpotent endomorphisms

([1], p.133-136)

In this section we will consider those linear endomorphisms $f \in \mathrm{End}_{\mathbb{C}}(V)$ whose only eigenvalue is 0. This necessarily implies that

$$\chi_f(\lambda) = \lambda^n.$$

We will see that for such endomorphisms there is a (ordered) basis $\mathcal{B}$ of V such that $[f]_{\mathcal{B}}$ is 'nearly diagonal'.

Definition 2.3.1. An endomorphism $f \in \mathrm{End}_{\mathbb{C}}(V)$ is called *nilpotent* if there exists $r \in \mathbb{N}$ such that $f^r = 0_{\mathrm{End}_{\mathbb{C}}(V)}$, so that $f^r(v) = 0_V$, for every $v \in V$.

A matrix $A \in Mat_n(\mathbb{C})$ is called *nilpotent* if the endomorphism $T_A \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ is nilpotent.

Lemma 2.3.2. *Let $f \in \mathrm{End}_{\mathbb{C}}(V)$ be a nilpotent endomorphism. Then, the only eigenvalue of f is $\lambda = 0$ so that $\chi_f(\lambda) = \lambda^{\dim V}$.*

Proof: Suppose that $v \in V$ is an eigenvector of f with associated eigenvalue λ. Therefore, we have $v \neq 0$ and $f(v) = \lambda v$. Suppose that $f^r = 0$. Then,

$$0 = f^r(v) = f \circ \cdots \circ f(v) = f \circ \cdots \circ f(\lambda v) = \lambda^r v.$$

Thus, as $v \neq 0$ we must have $\lambda^r = 0$ (Proposition 1.2.5) implying that $\lambda = 0$. □

For a nilpotent endomorphism f (resp. matrix $A \in Mat_n(\mathbb{C})$) we define the *exponent of f (resp. of A)*, denoted $\eta(f)$ (resp. $\eta(A)$), to be the smallest $r \in \mathbb{N}$ such that $f^r = 0$ (resp. $A^r = 0$). Therefore, if $\eta(f) = r$ then there exists $v \in V$ such that $f^{r-1}(v) \neq 0_V$.

For $v \in V$ we define the *height of v (with respect to f)*, denoted $\mathrm{ht}(v)$, to be the smallest integer m such that $f^m(v) = 0_V$, while $f^{m-1}(v) \neq 0_V$. Hence, for every $v \in V$ we have $\mathrm{ht}(v) \leq \eta(f)$.

Define $H_k = \{v \in V \mid \mathrm{ht}(v) \leq k\}$, the set of vectors that have height no greater than k; this is a subspace of V.[41]

Let $f \in \mathrm{End}_{\mathbb{C}}(V)$ be a nilpotent endomorphism. Then, we obviously have $H_{\eta(f)} = V$, $H_0 = \{0_V\}$ and a sequence of subspaces

$$\{0_V\} = H_0 \subset H_1 \subset \cdots \subset H_{\eta(f)-1} \subset H_{\eta(f)} = V.$$

Let us denote

$$\dim H_i = m_i,$$

so that we have

$$0 = m_0 \leq m_1 \leq ... \leq m_{\eta(f)-1} \leq m_{\eta(f)} = \dim V.$$

We are going to construct a basis of V: for ease of notation we let $\eta(f) = k$. Assume that $k \neq 1$, so that f is not the zero endomorphism of V.

1. Let G_k be a *complementary subspace* of H_{k-1} so that

$$H_k = H_{k-1} \oplus G_k,$$

   and let $(z_1, ..., z_{p_1})$ be an ordered basis of G_k. Then, since $z_j \in H_k \setminus H_{k-1}$ we have that $f^{k-1}(z_j) \neq 0_V$, for each j.

[40] This is a well-defined function since U_i is f-invariant.
[41] Exercise: show this.

2. Consider the vectors $f(z_1), f(z_2), \dots, f(z_{p_1})$. We have, for each j,

$$f^{k-1}(f(z_j)) = f^k(z_j) = 0_V, \quad \text{since } z_j \in H_k,$$

so that $f(z_j) \in H_{k-1}$, for each j. In addition, we can't have $f(z_j) \in H_{k-2}$, else

$$0_V = f^{k-2}(f(z_j)) = f^{k-1}(z_j),$$

implying that $z_j \in H_{k-1}$.

Moreover, the set $S_1 = \{f(z_1), f(z_2), \dots, f(z_{p_1})\} \subset H_{k-1} \setminus H_{k-2}$ is linearly independent: indeed, suppose that there is a linear relation

$$c_1 f(z_1) + \dots + c_{p_1} f(z_{p_1}) = 0_V.$$

with $c_1, \dots, c_{p_1} \in \mathbb{C}$. Then, since f is a linear morphism we obtain

$$f(c_1 z_1 + \dots + c_{p_1} z_{p_1}) = 0_V,$$

so that $c_1 z_1 + \dots + c_{p_1} z_{p_1} \in H_1 \subset H_{k-1}$.

Hence, we have $c_1 z_1 + \dots + c_{p_1} z_{p_1} \in H_{k-1} \cap G_k = \{0_V\}$, so that $c_1 z_1 + \dots + c_{p_1} z_{p_1} = 0_V$. Hence, because $\{z_1, \dots, z_{p_1}\}$ is linearly independent we must have $c_1 = \dots = c_{p_1} = 0 \in \mathbb{C}$. Thus, S_1 is linearly independent.

3. $\operatorname{span}_{\mathbb{C}} S_1 \cap H_{k-2} = \{0_V\}$: otherwise, we could find a linear combination

$$c_1 f(z_1) + \dots + c_{p_1} f(z_{p_1}) \in H_{k-2},$$

with some $c_i \neq 0$. Then, we would have

$$0_V = f^{k-2}(c_1 f(z_1) + \dots + c_{p_1} f(z_{p_1})) = f^{k-1}(c_1 z_1 + \dots + c_{p_1} z_{p_1}),$$

so that $c_1 z_1 + \dots + c_{p_1} z_{p_1} \in H_{k-1} \cap G_k = \{0_V\}$ which gives all $c_j = 0$, by linear independence of the z_j's. But this contradicts that some c_i is nonzero so that our initial assumption that $\operatorname{span}_{\mathbb{C}} S_1 \cap H_{k-2} \neq \{0_V\}$ is false.

Hence, we have

$$\operatorname{span}_{\mathbb{C}} S_1 + H_{k-2} = \operatorname{span}_{\mathbb{C}} S_1 \oplus H_{k-2} \subset H_{k-1}.$$

In particular, we see that $m_k - m_{k-1} \leq m_{k-1} - m_{k-2}$.

4. Let G_{k-1} be a complementary subspace of $H_{k-2} \oplus \operatorname{span}_{\mathbb{C}} S_1$ in H_{k-1}, so that

$$H_{k-1} = H_{k-2} \oplus \operatorname{span}_{\mathbb{C}} S_1 \oplus G_{k-1},$$

and let $(z_{p_1+1}, \dots, z_{p_2})$ be an ordered basis of G_{k-1}.

5. Consider the subset $S_2 = \{f^2(z_1), \dots, f^2(z_{p_1}), f(z_{p_1+1}), \dots, f(z_{p_2})\}$. Then, as in 2, 3, 4 above we have that

$$S_2 \subset H_{k-2} \setminus H_{k-3},$$

S_2 is linearly independent and $\operatorname{span}_{\mathbb{C}} S_2 \cap H_{k-3} = \{0_V\}$. Therefore, we have

$$\operatorname{span}_{\mathbb{C}} S_2 + H_{k-3} = \operatorname{span}_{\mathbb{C}} S_2 \oplus H_{k-3} \subset H_{k-2},$$

so that $m_{k-1} - m_{k-2} \leq m_{k-2} - m_{k-3}$.

6. Let G_{k-2} be a complementary subspace of $\text{span}_{\mathbb{C}} S_2 \oplus H_{k-3}$ in H_{k-2}, so that

$$H_{k-2} = H_{k-3} \oplus \text{span}_{\mathbb{C}} S_2 \oplus G_{k-2},$$

and $(z_{p_2+1}, \dots, z_{p_3})$ be an ordered basis of G_{k-2}.

7. Consider the subset $S_3 = \{f^3(z_1), \dots, f^3(z_{p_1}), f^2(z_{p_1+1}), \dots, f^2(z_{p_2}), f(z_{p_2+1}), \dots, f(z_{p_3})\}$. Again, it can be shown that

$$S_3 \subset H_{k-3} \setminus H_{k-4},$$

S_3 is linearly independent and $\text{span}_{\mathbb{C}} S_3 \cap H_{k-4} = \{0_V\}$. We obtain $m_{k-2} - m_{k-3} \leq m_{k-3} - m_{k-4}$.

8. Proceed in this fashion to obtain a basis of V. We denote the vectors we have obtained in a table

$$(2.3.1) \qquad \begin{array}{ccccccccccc} z_1, & \dots & z_{p_1}, & & & & & & & & \\ f(z_1), & \dots & f(z_{p_1}), & z_{p_1+1}, & \dots & z_{p_2}, & & & & & \\ \vdots & & & & \vdots & & & & & & \\ f^{k-1}(z_1), & \dots & f^{k-1}(z_{p_1}), & f^{k-2}(z_{p_1+1}), & \dots & f^{k-2}(z_{p_2}), & \dots & z_{p_{k-1}+1}, & \dots & z_{p_k}, & \end{array}$$

where the vectors in the i^{th} row have height $k - i + 1$, so that vectors in the last row have height 1.

Also, note that each column determines an f-invariant subspace of V, namely the span of the vectors in the column.

Lemma 2.3.3. *Let W_i denote the span of the i^{th} column of vectors in the table above. Set $p_0 = 1$. Then,*

$$\dim W_i = k - j, \quad \text{if } p_j + 1 \leq i \leq p_{j+1}.$$

Proof: Suppose that $p_j + 1 \leq i \leq p_{j+1}$. Then, we have

$$W_i = \text{span}_{\mathbb{C}}\{z_i, f(z_i), \dots, f^{k-j-1}(z_i)\}.$$

Suppose that there exists a linear relation

$$c_0 z_i + c_1 f(z_i) + \dots + c_{k-j-1} f^{k-j-1}(z_i) = 0_V.$$

Then, applying f^{k-j-1} to both sides of this equation gives

$$c_0 f^{k-j-1}(z_i) + c_1 f^{k-j}(z_i) + \dots + c_{k-j-1} f^{2k-2j-2}(z_i) = 0_V.$$

Now, as z_i has height $k-j$ (this follows because the vector at the top of the i^{th} column is in the $(k-j)^{th}$ row, therefore as height $(k-j)$) the previous equation gives

$$c_0 f^{k-j-1}(z_i) + 0_V + \dots + 0_V = 0_V,$$

so that $c_0 = 0$, since $f^{k-j-1}(z_i) \neq 0_V$. Thus, we are left with a linear relation

$$c_1 f(z_i) + \dots + c_{k-j-1} f^{k-j-1}(z_i) = 0_V,$$

and applying f^{j-k-2} to this equation will give $c_1 = 0$, since $f(z_i)$ has height $k - j - 1$. Proceeding in this manner we find that $c_0 = c_1 = \dots c_{j-k-1} = 0$ and the result follows. □

Thus, the information recorded in (2.3.1) and Lemma 2.3.3 proves the following

Theorem 2.3.4. *Let $f \in \text{End}_{\mathbb{C}}(V)$ be a nilpotent endomorphism with exponent $\eta(f) = k$. Then, there exists integers $d_1, \dots, d_k \in \mathbb{Z}_{\geq 0}$ so that*

$$kd_1 + (k-1)d_2 + \dots + 2d_{k-1} + 1d_k = \dim V,$$

and f-invariant subspaces

$$W_1^{(k)}, \dots, W_{d_1}^{(k)}, W_1^{(k-1)}, \dots, W_{d_2}^{(k-1)}, \dots, W_1^{(1)}, \dots, W_{d_k}^{(1)} \subset V,$$

with $\dim_{\mathbb{C}} W_i^{(j)} = j$*, such that*

$$V = W_1^{(k)} \oplus \cdots \oplus W_{d_1}^{(k)} \oplus W_1^{(k-1)} \oplus \cdots \oplus W_{d_2}^{(k-1)} \oplus \cdots \oplus W_1^{(1)} \oplus \cdots \oplus W_{d_k}^{(1)}.$$

Moreover, there is an ordered basis $\mathcal{B}_i^{(j)}$ *of* $W_i^{(j)}$ *such that*

$$[f_{|W_i^{(j)}}]_{\mathcal{B}_i^{(j)}} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}.$$

We call such matrices 0-Jordan blocks. *Hence, we can write the matrix of f relative to* $\mathcal{B} = \bigcup_{i,j} \mathcal{B}_i^{(j)}$ *as a block diagonal matrix for which all of the blocks are 0-Jordan blocks and are of nonincreasing size as we move from left to right.*

Moreover, the geometric multiplicity of 0 *as an eigenvalue of f is equal to the number of blocks of the matrix* $[f]_{\mathcal{B}}$ *and this number equals the sum*

$$d_1 + d_2 + \dots + d_k = \dim E_0.$$

Proof: Everything except for the final statement follows from the construction of the basis $\mathcal{B}$ made prior to the Theorem.

The last statement is shown as follows: we have that $E_0 = H_1$, so that the 0-eigenspace of f consists of the set of all height 1 vectors in V.[42] Moreover, the construction of the basis $\mathcal{B}$ shows that a basis of H_1 is given by the bottom row of the table (2.3.1) and that this basis has the size specified. □

Corollary 2.3.5. *Let* $A \in Mat_n(\mathbb{C})$ *be a nilpotent matrix. Then, A is similar to a block diagonal matrix for which all of the blocks are 0-Jordan blocks.*

Proof: Consider the endomorphism $T_A \in \text{End}_{\mathbb{C}}(\mathbb{C}^n)$ and apply Theorem 2.3.4. Then, we have a basis $\mathcal{B}$ such that $[T_A]_{\mathcal{B}}$ takes the desired form. Now, use Corollary 1.7.7 and $[T_A]_{\mathcal{S}^{(n)}} = A$ to deduce the result. □

Definition 2.3.6. Let $n \in \mathbb{N}$. A *partition of n* is a decomposition of n into a sum of positive integers. If we have a partition of n

$$n = n_1 + \dots + n_l, \quad \text{with } n_1, \dots, n_l \in \mathbb{N}, \quad n_1 \leq n_2 \leq \dots \leq n_l,$$

then we denote this partition

$$1^{r_1} 2^{r_2} \cdots n_l^{r_{n_l}},$$

where we are assuming that 1 appears r_1 times in the partition of n, 2 appears r_2 times etc.

For example, consider the partition of 13

$$13 = 1 + 1 + 1 + 2 + 4 + 4,$$

then we denote this partition

$$1^3 2^1 4^2.$$

[42] Check this.

For a nilpotent endomorphism $f \in \text{End}_{\mathbb{C}}(V)$ we define its *nilpotent class* to be the set of all nilpotent endomorphisms g of V for which there is some ordered basis $\mathcal{C} \subset V$ with

$$[f]_{\mathcal{B}} = [g]_{\mathcal{C}},$$

where $\mathcal{B}$ is the basis described in Theorem 2.3.4.

We define the *partition associated to the nilpotent class of* f, denoted $\pi(A)$, to be the partition $1^{d_k}2^{d_{k-1}} \cdots k^{d_1}$ obtained in Theorem 2.3.4. We will also call this partition the *partition associated to* f.

For a matrix $A \in Mat_n(\mathbb{C})$ we define its nilpotent class (or *similarity class*) to be the nilpotent class of the endomorphism T_A. We define the *partition associated to* A to be the partition associated to T_A.

Theorem 2.3.7 (Classification of nilpotent endomorphisms)**.** *Let $f, g \in \text{End}_{\mathbb{C}}(V)$ be nilpotent endomorphisms of V. Then, f and g lie in the same nilpotent class if and only if the partitions associated to f and g coincide.*

Corollary 2.3.8. *Let $A, B \in Mat_n(\mathbb{C})$ be nilpotent matrices. Then, f and g are similar if and only if the partitions associated to A and B coincide.*

Proof: We simply note that if T_A and T_B are in the same nilpotent class then there are bases $\mathcal{B}, \mathcal{C} \subset \mathbb{C}^n$ such that

$$[T_A]_{\mathcal{B}} = [T_B]_{\mathcal{C}}.$$

Hence, if $P_1 = P_{\mathcal{S}^{(n)} \leftarrow \mathcal{B}}, P_2 = P_{\mathcal{S}^{(n)} \leftarrow \mathcal{C}}$ then we must have

$$P_1^{-1}AP_1 = P_2^{-1}BP_2,$$

so that

$$P_2P_1^{-1}AP_1P_2^{-1} = B.$$

Now, since $P_2P_1^{-1} = (P_1P_2^{-1})^{-1}$ we have that A and B are similar precisely when T_A and T_B are in the same nilpotent class. The result follows. □

2.3.1 Determining partitions associated to nilpotent endomorphisms

Given a nilpotent endomorphism $f \in \text{End}_{\mathbb{C}}(V)$ (or nilpotent matrix $A \in Mat_n(\mathbb{C})$) how can we determine the partition associated to f (resp. A)?

Once we have chosen an ordered basis $\mathcal{B}$ of V we can consider the nilpotent matrix $[f]_{\mathcal{B}}$. Then, the problem of determining the partition associated to f reduces to determining the partition associated to $[f]_{\mathcal{B}}$. As such, we need only determine the partition associated to a nilpotent matrix $A \in Mat_n(\mathbb{C})$.

1. Determine the exponent of A, $\eta(A)$, by considering the products A^2, A^3, etc. The first r such that $A^r = 0$ is the exponent of A.

2. We can determine the subspaces H_i since

$$H_i = \{\underline{x} \in \mathbb{C}^n \mid \text{ht}(\underline{x}) \leq i\} = \ker T_{A^i}.$$

   In particular, we have that $\dim H_i$ is the number of non-pivot columns of A^i.

3. $d_1 = \dim H_{\eta(A)} - \dim H_{\eta(A)-1}$.

4. $d_2 = \dim H_{\eta(A)-1} - \dim H_{\eta(A)-2} - d_1$.

5. $d_3 = \dim H_{\eta(A)-2} - \dim H_{\eta(A)-3} - d_2$.

6. Thus, we can see that $d_i = \dim H_{\eta(A)-(i-1)} - \dim H_{\eta(A)-i} - d_{i-1}$, for $1 \leq i \leq \eta(A)$.

Hence, the partition associated to A is

$$\pi(A): \ 1^{d_{\eta(A)}}2^{d_{\eta(A)-1}} \cdots \eta(A)^{d_1}.$$

Example 2.3.9. Consider the endomorphism

$$f : \mathbb{C}^5 \to \mathbb{C}^5 \; ; \; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ 0 \\ x_4 \\ 0 \\ 0 \end{bmatrix}.$$

Then, with respect to the standard basis $\mathcal{S}^{(5)}$ we have that

$$A \stackrel{def}{=} [f]_{\mathcal{S}^{(5)}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

You can check that $A^2 = 0$ so that $\eta(A) = 2$. Then,

- $d_1 = \dim H_2 - \dim H_1 = 5 - 3 = 2$, since $H_1 = \ker T_A$ has dimension 3 (there are 3 non-pivot columns of A).
- $d_2 = \dim H_1 - \dim H_0 - d_1 = 3 - 0 - 2 = 1$, since $H_0 = \{0\}$.

Hence, the partition associated to A is

$$\pi(A) : \; 12^2 \leftrightarrow 1 + 2 + 2 = 5;$$

there are three 0-Jordan blocks - two of size 2 and one of size 1.

You can check that the following matrix B is nilpotent

$$B = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

and that the partition associated to B is

$$\pi(B) : \; 1^3 2 \leftrightarrow 1 + 1 + 1 + 2 = 5$$

- We have $B^2 = 0$ so that $\eta(B) = 2$.
- $d_1 = \dim H_2 - \dim H_1 = 5 - 4 = 1$, since $H_1 = \ker T_B$ has dimension 4 (there are 4 non-pivot columns of B).
- $d_2 = \dim H_1 - \dim H_0 - d_1 = 4 - 0 - 1 = 3$, since $H_0 = \{0\}$.

Thus, A and B are not similar, by Corollary 2.3.8. However, since the matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

has associated partition

$$\pi(C) : \; 1^3 2,$$

then we see that B is similar to C, by Corollary 2.3.8.

Moreover, there are four 0-Jordan blocks of B (and C) - one of size 2 and three of size 1.