*Proof:* This is an exercise in row-reduction and one which you should already be familiar with.

Recall that for any linear morphism $f : \mathbb{K}^n \to \mathbb{K}^m$, there is a matrix $A_f$ called the *standard matrix associated to* $f$ such that

$$\text{for every } \underline{x} \in \mathbb{K}^n, \ f(\underline{x}) = A_f \underline{x}.$$

$A_f$ is defined to be the $m \times n$ matrix whose $i^{th}$ column is the column vector $f(e_i)$, where $e_i$ is the $i^{th}$ standard basis vector of $\mathbb{K}^n$ (Example 1.2.6).
Then, it will be an exercise to show the following:

- $f$ is injective if and only if $A_f$ has a pivot in every column, and

- $f$ is surjective if and only if $A_f$ has a pivot in every row.

Therefore, since we are assuming that $f$ is an isomorphism it must, by definition, be a bijective morphism. Hence, it is both injective and surjective. By the preceding comments we must therefore have a pivot in every column and every row. The only way that this can happen is if $n = m$. □

We will see later, after the introduction of bases for vector spaces, that the converse if also true: namely, **if $n = m$ then $\mathbb{K}^n$ and $\mathbb{K}^m$ are isomorphic.**

**Proposition 1.4.12.** *Let $V, W$ be $\mathbb{K}$-vector spaces, $E \subset V$ a subset of $V$. Let $f : V \to W$ be an isomorphism from $V$ to $W$ and denote $f(E) = \{f(e) \mid e \in E\}$, the* image set *of $E$.*[24] *Then,*

- *$E$ is linearly independent if and only if $f(E)$ is linearly independent.*

- *$E$ spans $V$ if and only if $f(E)$ spans $W$.*

*Proof:* Left to the reader. □

## 1.5 Bases, Dimension

In this section we will introduce the notion a *basis* of a $\mathbb{K}$-vector space. We will provide several equivalent approaches to the definition of a basis and see that the size of a basis is an invariant[25] of a $\mathbb{K}$-vector space which we will call its *dimension*. You should have already seen the words *basis* and *dimension* in your previous linear algebra course so do not abandon what you already know! We are just simply going to provide some interesting(?) ways we can think about a basis; in particular, these new formulations will allow us to extend our results to infinite dimensional vector spaces.

First, we must introduce a (somewhat annoying) idea to keep us on the straight and narrow when we are considering bases, that of an *ordered set*.

**Definition 1.5.1** (Ordered Set)**.** An *ordered set* is a nonempty set $S$ for which we have provided a 'predetermined ordering' on $S$.

**Remark 1.5.2.** 1. This definition might seem slightly confusing (and absurd); indeed, it is both of these things as I have not rigorously defined what a 'predetermined ordering' is. Please don't dwell too much on this as we will only concern ourselves with orderings of finite sets (for which it is easy to provide an ordering) or the standard ordering of $\mathbb{N}$. An ordered set is literally just a nonempty set $S$ whose elements have been (strictly) ordered in some way.

For example, suppose that $S = [3] = \{1, 2, 3\}$. We usually think of $S$ has having its natural ordering $(1, 2, 3)$. However, when we consider this ordering we are actually considering the ordered set $(1, 2, 3)$ and not the set $S$... Confused? I thought so. We could also give the objects in $S$ the ordering $(2, 1, 3)$ and when we do this we have a defined a *different* ordered set to $(1, 2, 3)$.

[24] This is not necessarily the same as the *image of f*, im$f$, introduced before.

[25] In mathematics, when we talk of an *invariant* we usually mean an atrribute or property of an object that remains unchanged whenever that object is transformed to another via an isomorphism (in an appropriate sense). For example, you may have heard of the *genus* of a (closed) geometric surface: this is an invariant of a surface that counts the number of 'holes' that exist within a (closed) surface. Perhaps you have heard or read the phrase that a mathematician thinks a coffee mug and a donut are indistuingishable. This is because we can continuously deform a donut into a coffee mug, and vice versa. This continuous deformation can be regarded as an 'isomorphism' in the world of (closed) geometric surfaces.

If you are still confused, do not worry. Here is another example: consider the set

$$S = \{\text{Evans Hall, Doe Library, Etcheverry Hall}\}.$$

Now, there is no predetermined way that we can order this set: I might choose the ordering

$$(\text{Evans Hall, Etcheverry Hall, Doe Library}),$$

whereas you might think it better to choose an ordering

$$(\text{Doe Library, Etcheverry Hall, Evans Hall}).$$

Of course, neither of these choices of orderings is 'right' and we are both entitled to our different choices. However, these ordered sets **are different**.

The reason we require this silly idea is when we come to consider coordinates (with respect to a given ordered basis). Then, it will be extremely important that we declare an ordering of a basis and that we are consistent with this choice.

2. Other examples of ordered sets include $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ with their usual orderings. We can also order $\mathbb{C}$ in an ordering called a *lexicographic ordering*: here we say that $z = a_1 + b_1\sqrt{-1} < w = a_2 + b_2\sqrt{-1}$ if and only if either, $a_1 < a_2$, or, $a_1 = a_2$ *and* $b_1 < b_2$. Think of this as being similar to the way that words are ordered in the dictionary, except now we consider only 'words' consisting of two 'letters', each of which is a real number.

3. What about some really bizarre set that might be infinite; for example, $\mathbb{R}^{\mathbb{R}}$, the set of all functions $\mathbb{R} \to \mathbb{R}$. How can we order this set? In short, I have no idea! However, there are some very deep results from mathematical logic that say that, if we assume a certain axiom of mathematics (the so-called Axiom of Choice), then every set can be ordered in some manner. In fact, it has been shown that the Axiom of Choice of logically equivalent to this ordering property of sets! If you want to learn more then you should consult Wikipedia and take Math 125A in the Fall Semester.[26]

Therefore, no matter how weird or massively infinite a set is, if you are assuming the Axiom of Choice (which we are) then you can put an ordering on that set, even though you will (a priori) have no idea what that ordering is! All that matters is that such an ordering **exists**.

**Definition 1.5.3** (Basis; Ordered Basis)**.** Let $V$ be a $\mathbb{K}$-vector space. A nonempty subset $\mathcal{B} \subset V$ is called a ($\mathbb{K}$)-*basis of* $V$ if

- $\mathcal{B}$ is linearly independent (over $\mathbb{K}$), and
- if $\mathcal{B} \subset \mathcal{B}'$ and $\mathcal{B}'$ is linearly independent (over $\mathbb{K}$), then $\mathcal{B}' = \mathcal{B}$.

In this case, we say that $\mathcal{B}$ is *maximal linearly independent*.

An *ordered* ($\mathbb{K}$)-*basis of* $V$ is a ($\mathbb{K}$)-basis of $V$ that is an ordered set.

**Remark 1.5.4.** 1. You may have seen a basis of a $\mathbb{K}$-vector space $V$ defined as a subset $\mathcal{B} \subset V$ such that $\mathcal{B}$ is linearly independent (over $\mathbb{K}$) and such that $\text{span}_{\mathbb{K}} \mathcal{B} = V$. The definition given above is equivalent to this and it has been used as the definition of a basis to encapsulate the intuition behind a basis: namely, if $\mathbb{K} = \mathbb{R}$, we can think of a basis of an $\mathbb{R}$-vector space as a choice of 'independent directions' that allows us to consider well-defined coordinates. This idea of 'independent directions' is embodied in the fact that a basis must be a linearly independent set; and the assumption of maximal linear independence is what allows us to obtain well-defined coordinates.

However, just to keep our minds at ease our next result will show the equivalence between Definition 1.5.3 and the definition you have probably seen before.

[26] I have to admit that I do not know any mathematical logic but have come across these ideas during my own excursions in mathematics. There are lots of many interesting results that can be obtained if one assumes the Axiom of Choice: one is called the Banach-Tarski Paradox; another, which is directly related to our studies, is the existence of a basis for *any* $\mathbb{K}$-vector space. In fact, the Axiom of Choice is logically equivalent to the existence of a basis for any $\mathbb{K}$-vector space.

2. We will also see in the homework that we can consider a basis to be a *minimal spanning set* (in an appropriate sense to be defined later); this is recorded in Proposition 1.5.9.

3. It is important to remember that a basis is a subset of $V$ and **not** a subspace of $V$.

4. We will usually not call a basis of a $\mathbb{K}$-vector space a '$\mathbb{K}$-basis', it being implicitly assumed that we are considering only $\mathbb{K}$-bases when we are talking about $\mathbb{K}$-vector spaces. As such, we will only use the terminology 'basis' from now on.

**Proposition 1.5.5.** *Let $V$ be a $\mathbb{K}$-vector space and $\mathcal{B} \subset V$ a basis of $V$. Then, $\text{span}_{\mathbb{K}} \mathcal{B} = V$. Conversely, if $\mathcal{B} \subset V$ is a linearly independent spanning set of $V$, then $\mathcal{B}$ is a basis of $V$*

*Proof:* Let us denote $W = \text{span}_{\mathbb{K}} \mathcal{B}$. Then, because $\mathcal{B} \subset V$ we have $W \subset V$. To show that $W = V$ we are going to assume otherwise and obtain a contradiction. So, suppose that $W \neq V$. This means that there exists $v_0 \in V$ such that $v_0 \notin W$. In particular, $v_0 \notin \mathcal{B} \subset W$. Now, consider the subset $\mathcal{B}' = \mathcal{B} \cup \{v_0\} \subset V$.

Then, by Corollary 1.3.5, $\mathcal{B}'$ is linearly independent.

Now, we use the maximal linear independence property of $\mathcal{B}$: since $\mathcal{B} \subset \mathcal{B}'$ and $\mathcal{B}'$ is linearly independent we must have $\mathcal{B}' = \mathcal{B}$, because $\mathcal{B}$ is a basis. Hence, $v_0 \in \mathcal{B}$. But this contradicts that fact that $v_0 \notin \mathcal{B}$. Therefore, our intial assumption, that $W \neq V$, must be false and we must necessarily have $W = V$.

Conversely, suppose that $\mathcal{B}$ is a linearly independent subset of $V$ such that $\text{span}_{\mathbb{K}} \mathcal{B} = V$. We want to show that $\mathcal{B}$ is a basis, so we must show that $\mathcal{B}$ satisfies the maximal linearly independent property of Definition 1.5.3.

Therefore, suppose that $\mathcal{B} \subset \mathcal{B}'$ and that $\mathcal{B}'$ is linearly independent; we must show that $\mathcal{B}' = \mathcal{B}$. Now, since $\mathcal{B} \subset \mathcal{B}'$ we have $V = \text{span}_{\mathbb{K}} \mathcal{B} \subset \text{span}_{\mathbb{K}} \mathcal{B}' \subset V$, using Lemma 1.3.9. Hence, $\text{span}_{\mathbb{K}} \mathcal{B}' = V = \text{span}_{\mathbb{K}} \mathcal{B}$. Assume that $\mathcal{B} \neq \mathcal{B}'$; we aim to provide a contradiction. Then, for each $w \in \mathcal{B}' \setminus \mathcal{B}$ we have $w \in \text{span}_{\mathbb{K}} \mathcal{B}' = \text{span}_{\mathbb{K}} \mathcal{B}$, so that there exists an expression

$$w = \lambda_1 b_1 + ... + \lambda_n b_n,$$

where $b_1, ..., b_n \in \mathcal{B}$. But this means that we have a nontrivial[27] linear relation among vectors in $\mathcal{B}'$ (recall that, as $\mathcal{B} \subset \mathcal{B}'$, we have $b_1, ..., b_n \in \mathcal{B}'$). However, $\mathcal{B}'$ is linearly independent so that no such nontrivial linear relation can exist. Hence, our initial assumption of the existence of $w \in \mathcal{B}' \setminus \mathcal{B}$ is false, so that $\mathcal{B}' = \mathcal{B}$. The result follows. □

**Corollary 1.5.6.** *Let $V$ be a $\mathbb{K}$-vector space, $\mathcal{B} \subset V$ a basis of $V$. Then, for every $v \in V$ there exists a unique expression*

$$v = \lambda_1 b_1 + ... + \lambda_n b_n,$$

*where $b_1, ..., b_n \in \mathcal{B}$, $\lambda_1, ..., \lambda_n \in \mathbb{K}$, $n \in \mathbb{N}$.*

*Proof:* By Proposition 1.5.5, we have that $\text{span}_{\mathbb{K}} \mathcal{B} = V$ so that, for every $v \in V$, we can write $v$ as a linear combination of vectors in $\mathcal{B}$

$$v = \lambda_1 b_1 + ... + \lambda_n b_n, \quad b_1, ..., b_n \in \mathcal{B},$$

where we can further assume that none of $\lambda_1, ..., \lambda_n$ is equal to zero.

We need to show that this expression is *unique*: so, suppose that we can write $v$ as a different linear combination

$$v = \mu_1 b_1' + ... + \mu_k b_k', \quad b_1', ..., b_k' \in \mathcal{B},$$

again assuming that none of the $\mu_1, ..., \mu_k$ are equal to zero.

Therefore, we have

$$\lambda_1 b_1 + ... + \lambda_n b_n = v = \mu_1 b_1' + ... + \mu_k b_k',$$

giving a linear relation

$$\lambda_1 b_1 + ... + \lambda_n b_n - (\mu_1 b_1' + ... + \mu_k b_k') = 0_V.$$

[27] Why is this linear relation nontrivial?

Thus, since $\mathcal{B}$ is linearly independent this linear relation must be trivial and, furthermore, since we have assumed that none of the $\lambda$'s or $\mu$'s are zero, the only way that this can happen is if $n = k$ and, without loss of generality, $b_i = b'_i$ and $\lambda_i = \mu_i$. Hence, the linear combination given above is unique. □

**Corollary 1.5.7.** *Let $V$ be a $\mathbb{K}$-vector space, $\mathcal{B} = (b_1, \dots, b_n) \subset V$ an ordered basis containing finitely many vectors. Then, $V$ is isomorphic to $\mathbb{K}^n$.*

*Proof:* This is just a simple restatement of Corollary 1.5.6: we define a function

$$[-]_{\mathcal{B}} : V \to \mathbb{K}^n \ ; \ v \mapsto [v]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix},$$

where

$$v = \lambda_1 b_1 + \dots + \lambda_n b_n,$$

is the unique expression for $v$ coming from Corollary 1.5.6. Uniqueness shows that $[-]_{\mathcal{B}}$ is indeed a well-defined function.

It will be left to the reader to show that $[-]_{\mathcal{B}}$ is a bijective $\mathbb{K}$-linear morphism, thereby showing that it is an isomorphism. □

**Definition 1.5.8.** Let $V$ be a $\mathbb{K}$-vector space, $\mathcal{B} = \{b_1, \dots, b_n\} \subset V$ an ordered basis containing finitely many vectors. Then, the linear morphism

$$[-]_{\mathcal{B}} : V \to \mathbb{K}^n \ ; \ v \mapsto [v]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix},$$

introduced above is called the *$\mathcal{B}$-coordinate map* or *$\mathcal{B}$-coordinate morphism*.

The following Proposition provides yet another viewpoint of the idea of a basis: it says that a basis is a spanning set that satisfies a certain minimality condition.

**Proposition 1.5.9.** *Let $V$ be a $\mathbb{K}$-vector space, $\mathcal{B} \subset V$ a basis of $V$. Then, $\mathcal{B}$ is a minimal spanning set - namely,*

- $\text{span}_{\mathbb{K}} \mathcal{B} = V$, *and*
- *if $\mathcal{B}' \subset \mathcal{B}$ is such that $\text{span}_{\mathbb{K}} \mathcal{B}' = V$ then $\mathcal{B}' = \mathcal{B}$.*

*A proof of this Proposition will appear as a homework exercise.*

Despite all of these results on bases of vector spaces we have still yet to give the most important fact concerning a basis: that a basis **exists** in an arbitrary $\mathbb{K}$-vector space.

The proof of the general case requires the use of a particularly subtle lemma, called Zorn's Lemma. You can read about Zorn's Lemma on Wikipedia and there you will see that Zorn's Lemma is equivalent to the Axiom of Choice (although the proof of this fact is quite difficult). You will also read on Wikipedia that Zorn's Lemma is logically equivalent to the existence of a basis for an arbitrary $\mathbb{K}$-vector space.

**Theorem 1.5.10.** *Let $V$ be a $\mathbb{K}$-vector space. Then, there exists a basis $\mathcal{B} \subset V$ of $V$.*

*Proof: Case 1:* There exists a finite subset $E \subset V$ such that $\text{span}_{\mathbb{K}} E = V$.

In this case we will use the Elimination Lemma (Lemma 1.3.10) to remove vectors from $E$ until we obtain a linearly independent set. Now, if $E$ is linearly independent then $E$ is a linearly independent spanning set of $V$ and so, by Proposition 1.5.5, $E$ is a basis of $V$. Therefore, assume that $E$ is linearly dependent. Then, if we write $E$ as an ordered set $E = \{e_1, \dots, e_n\}$, we can use Lemma 1.3.10 to remove a vector from $E$ so that the resulting set is also a spanning set of $V$; WLOG, we can assume that the vector we remove is $e_n$. Then, define $E^{(n-1)} = E \setminus \{e_n\}$ so that we have $\text{span}_{\mathbb{K}} E^{(n-1)} = V$. If $E^{(n-1)}$ is linearly independent then it must be a basis (as it is also a spanning set). If $E^{(n-1)}$ is linearly dependent then we can again use Lemma 1.3.10 to remove a vector from $E^{(n-1)}$ so that the resulting set is a spanning set of $V$; WLOG, we can assume that the vector we remove is $e_{n-1}$. Then, define $E^{(n-2)} = E \setminus \{e_{n-2}\}$ so that we have $\text{span}_{\mathbb{K}} E^{(n-2)} = V$. Proceeding in a similar fashion as before we will either have that $E^{(n-2)}$ is linearly independent (in which case it is a basis) or it will be linearly dependent and we can proceed as before, removing a vector to obtain a new set $E^{(n-3)}$ etc.

Since $E$ is a finite set this procedure must terminate after finitely many steps. The stage at which it terminates will have a produced a linearly independent spanning set of $V$, that is, a basis of $V$ (by Proposition 1.5.5).

*Case 2:* There does not exist a finite spanning set of $V$.

In this case we must appeal to Zorn's Lemma: basically, the idea is that we will find a basis by considering a maximal linearly independent subset of $V$. Zorn's Lemma is a technical result that allows us to show that such a subset always exists and therefore, by definition, must be a basis of $V$. □

**Theorem 1.5.11** (Basis Theorem)**.** *Let $V$ be a $\mathbb{K}$-vector space and $\mathcal{B} \subset V$ a basis such that $\mathcal{B}$ has only finitely many vectors. Then, if $\mathcal{B}'$ is another basis of $V$ then $\mathcal{B}'$ has the same number of vectors as $\mathcal{B}$.*

*Proof:* Let $\mathcal{B} = \{b_1, \dots, b_n\}$ and $\mathcal{B}' = \{b'_1, \dots, b'_m\}$ be two distinct bases of $V$. Then, by Corollary 1.5.7, we have isomorphisms

$$[-]_{\mathcal{B}} : V \to \mathbb{K}^n, \quad \text{and} \quad [-]_{\mathcal{B}'} : V \to \mathbb{K}^m.$$

Hence, we obtain an isomorphism (since the composition of two isomorphisms is again an isomorphism, by Lemma 0.2.4)

$$[-]_{\mathcal{B}'} \circ [-]_{\mathcal{B}}^{-1} : \mathbb{K}^n \to \mathbb{K}^m,$$

where $[-]_{\mathcal{B}'}^{-1} : \mathbb{K}^m \to V$ is the inverse morphism of $[-]_{\mathcal{B}'}$. Thus, using Theorem 1.4.11, we must have $n = m$, so that $\mathcal{B}$ and $\mathcal{B}'$ have the same size. □

Theorem 1.5.11 states that if $V$ is a $\mathbb{K}$-vector space admitting a finite basis $\mathcal{B}$, then every other basis of $V$ must have the same size as the set $\mathcal{B}$.

**Definition 1.5.12.** Let $V$ be a $\mathbb{K}$-vector space, $\mathcal{B} \subset V$ a basis of $V$ containing finitely many vectors. Then, the size of $\mathcal{B}$, $|\mathcal{B}|$, is called the *dimension of $V$ (over $\mathbb{K}$)* and is denoted $\dim_{\mathbb{K}} V$, or simply $\dim V$ when no confusion can arise. In this case we will also say that $V$ is *finite dimensional*. If $V$ is a $\mathbb{K}$-vector space that does not admit a finite basis then we will say that $V$ is *infinite dimensional*.

The Basis Theorem (Theorem 1.5.11) ensures that the dimension of a $\mathbb{K}$-vector space is a well-defined number (ie, it doesn't change when we choose a different basis of $V$).

Now that we have introduced the notion of dimension of a $\mathbb{K}$-vector space we can give one of the fundamental results of finite dimensional linear algebra.

**Theorem 1.5.13.** *Let $V$ and $W$ be $\mathbb{K}$-vector spaces such that $\dim_{\mathbb{K}} V = \dim_{\mathbb{K}} W < \infty$ is finite. Then, $V$ is isomorphic to $W$.*

This result, in essence, classifies all finite dimensional $\mathbb{K}$-vector spaces by their dimension. It tells us that any linear algebra question we can ask in a $\mathbb{K}$-vector space $V$ (for example, a question concerning linear independence or spans) can be translated to another $\mathbb{K}$-vector space $W$ which we know has the same dimension as $V$. This follows from Proposition 1.4.12.

This principle underlies our entire approach to finite dimensional linear algebra: given a $\mathbb{K}$-vector space $V$ such that $\dim_{\mathbb{K}} V = n$, Theorem 1.5.13 states that $V$ is isomorphic to $\mathbb{K}^n$ and Corollary 1.5.7 states that, once we have a basis $\mathcal{B}$ of $V$, we can use the $\mathcal{B}$-coordinate morphism as an isomorphism from $V$ to $\mathbb{K}^n$. Of course, we still need to find a basis! We will provide an approach to this problem after we have provided the (simple) proof of Theorem 1.5.13.

*Proof:* The statement that $V$ and $W$ have the same dimension is just saying that any basis of these vector spaces have the same number of elements. Let $\mathcal{B} \subset V$ be a basis of $V$, $\mathcal{C} \subset W$ a basis of $W$. Then, we have the coordinate morphisms

$$[-]_{\mathcal{B}} : V \to \mathbb{K}^n \quad \text{and} \quad [-]_{\mathcal{C}} : W \to \mathbb{K}^n,$$

both of which are isomorphisms. Then, the morphism

$$[-]_{\mathcal{C}}^{-1} \circ [-]_{\mathcal{B}} : V \to W,$$

is an isomorphism between $V$ and $W$. □

**Example 1.5.14.** 1. The ordered set $\mathcal{B} = (e_1, \dots, e_n) \subset \mathbb{K}^n$ is an ordered basis of $\mathbb{K}^n$, where $e_i$ is the column vector with a 1 in the $i^{th}$ entry and 0 elsewhere.

   **We will denote this basis $\mathcal{S}^{(n)}$.**

   It is easy to show that $\mathcal{S}^{(n)}$ is linearly independent and that $\text{span}_{\mathbb{K}} \mathcal{S}^{(n)} = \mathbb{K}^n$. Hence, we have that $\dim_{\mathbb{K}} \mathbb{K}^n = n$.

2. Let $S$ be a finite set and denote $S = \{s_1, \dots, s_k\}$. Then, $\mathcal{B} = (e_{s_1}, \dots, e_{s_k})$ is an ordered basis of $\mathbb{K}^S$, where $e_{s_i}$ is the elementary functions defined in Example 1.2.6.

   We have that $\mathcal{B}$ is linearly independent: for, if there is a linear relation

$$c_1 e_{s_1} + \dots + c_k e_{s_k} = 0_{\mathbb{K}^S},$$

   then, in particular, evaluating both sides of this equation (of functions) at $s_i$ gives

$$c_i = c_1 e_{s_1}(s_i) + \dots + c_k e_{s_k}(s_i) = (c_1 e_{s_1} + \dots + c_k e_{s_k})(s_i) = 0_{\mathbb{K}^S}(s_i) = 0.$$

   Hence, $c_i = 0$, for every $i$, and $\mathcal{B}$ is linearly independent.

   Furthermore, $\mathcal{B}$ is a spanning set of $\mathbb{K}^S$: let $f \in \mathbb{K}^S$. Then, we have an equality of functions

$$f = f(s_1)e_{s_1} + f(s_2)e_{s_2} + \dots + f(s_n)e_{s_n},$$

   which can be easily checked by showing that

$$f(t) = (f(s_1)e_{s_1} + f(s_2)e_{s_2} + \dots + f(s_n)e_{s_n})(t), \ \forall t \in S.$$

   Hence, $f \in \text{span}_{\mathbb{K}} \mathcal{B}$ so that, since $f$ was arbitrary, we find $\text{span}_{\mathbb{K}} \mathcal{B} = \mathbb{K}^S$.

   Hence, we see that $\dim_{\mathbb{K}} \mathbb{K}^S = |S|$.

3. It is not true that if $S$ is an infinite set then $\mathcal{B} = \{e_s \mid s \in S\}$ is a basis of $\mathbb{K}^S$, even though $\mathcal{B}$ is a linearly independent set. This is discussed in a worksheet.

4. As a particular example of 2 above, we see that $Mat_{m,n}(\mathbb{K})$ has as a basis the elementary matrices $\mathcal{B} = \{e_{ij} \mid (i,j) \in \{1, \dots, m\} \times \{1, \dots, n\}\}$. These are those matrices that have 0s for all entries except for a 1 in the $ij$-entry.

   Hence, we see that $\dim_{\mathbb{K}} Mat_{m,n}(\mathbb{K}) = mn$.

### 1.5.1 Finding a basis

In this section we will provide criteria for determining when a subset $E$ of a finite dimensional $\mathbb{K}$-vector space $V$ is a basis. Hopefully, this is just a recollection of results that you have seen before in your first linear algebra course.

Throughout this section we will fix a finite dimensional $\mathbb{K}$-vector space $V$ such that $\dim_{\mathbb{K}} V = n$ and an ordered basis $\mathcal{B} = (b_1, \dots, b_n)$ (which we know exists by Theorem 1.5.10).

**Proposition 1.5.15.** *Let $E \subset V$ be a nonempty subset of $V$.*

*a) If $E$ is linearly independent, then $|E| \leq n$.*

*b) If* $\operatorname{span}_{\mathbb{K}} E = V$*, then* $|E| \geq n$*,*

*c) If* $E \subset V$ *is linearly independent and* $F \subset V$ *is a spanning set, so that* $\operatorname{span}_{\mathbb{K}} F = V$*, then either* $k = n$ *and* $E$ *is a basis of* $V$*; or,* $E$ *can be extended to a basis of* $V$ *by adding to* $E$ *vectors from* $F$*. This means, if* $E = \{e_1, \dots, e_k\}$ *then we can find* $f_{k+1}, \dots, f_n \in F$ *such that* $\{e_1, \dots, e_k, f_{k+1}, \dots, f_n\}$ *is a basis of* $V$*.*

*Proof:* *a*) Suppose that $E$ is linearly independent, finite and nonempty and that $|E| > n$, say $|E| = k > n$ and denote $E = \{e_1, \dots, e_k\}$; we aim to provide a contradiction.

In this case, $E$ can't be a basis of $V$, for otherwise we would contradict the Basis Theorem (Theorem 1.5.11), as $E$ does not have $n$ vectors. Hence, since $E$ is linearly independent we must have that $\operatorname{span}_{\mathbb{K}} E \neq V$ (otherwise $E$ would be a basis, by Proposition 1.5.5). Moreover, we can't have $\mathcal{B} \subset \operatorname{span}_{\mathbb{K}} E$ as then we would have $V = \operatorname{span}_{\mathbb{K}} \mathcal{B} \subset \operatorname{span}_{\mathbb{K}} E$ implying that $V = \operatorname{span}_{\mathbb{K}} E$ (because we would have $\operatorname{span}_{\mathbb{K}} E \subset V$ and $V \subset \operatorname{span}_{\mathbb{K}} E$). Therefore, we can assume, without loss of generality, that $b_1 \notin \operatorname{span}_{\mathbb{K}} E$ so that, by the Elimination Lemma (Lemma 1.3.10), we have that $E_1 = E \cup \{b_1\}$ is a linearly independent set. Then, we can't have that $\operatorname{span}_{\mathbb{K}} E_1 = V$, else we would contradict the Basis Theorem. Thus, $\operatorname{span}_{\mathbb{K}} E_1 \neq V$. Now, without loss of generality, we can assume that $b_2 \notin \operatorname{span}_{\mathbb{K}} E_1$; otherwise, $b_2, \dots, b_n \in \operatorname{span}_{\mathbb{K}} E_1$ and $b_1 \in \operatorname{span}_{\mathbb{K}} E_1$, so that $\mathcal{B} \subset \operatorname{span}_{\mathbb{K}} E_1$ giving $V = \operatorname{span}_{\mathbb{K}} E_1$. Denote $E_2 = E_1 \cup \{b_2\}$. Then, again by the Elimination Lemma, we have that $E_2$ is a linearly independent set such that $\operatorname{span}_{\mathbb{K}} E_2 \neq V$ (else we would contradict the Basis Theorem). Proceeding in this way we obtain subsets

$$E_i = E_{i-1} \cup \{b_i\}, \ i = 1, \dots, n, \quad \text{with } E_0 \stackrel{def}{=} E,$$

that are linearly independent. In particular, we obtain the subset $E_n = E \cup \mathcal{B}$ that is linearly independent and strictly contains $\mathcal{B}$, contradicting the maximal linearly independent property of a basis. Therefore, our initial assumption that $|E| > n$ must be false, so that $|E| \leq n$.

If $E$ is infinite, then every subset of $E$ is linearly independent. Hence, we can find arbitrarily large linearly independent finite subsets of $E$. Choose a subset $E'$ such that $|E'| > n$. Then we are back in the previous situation, which we have just cannot hold. Hence, we can't have that $E$ is infinite.

*b*) This is consequence of the method of proof for Case 1 of Theorem 1.5.10. Indeed, either $E$ is an infinite set and there is nothing to prove, or $E$ is a finite set. Then, as in the proof of Theorem 1.5.10, we can find a basis $E' \subset E$ contained in $E$. Hence, by the Basis Theorem, we see that $n = |E'| \leq |E|$.

*c*) Let $E \subset V$ be a linearly independent subset of $V$. Then, by *a*) we know that $|E| \leq n$. Let us write $E = \{e_1, \dots, e_k\}$, so that $k \leq n$.

*Case 1:* $k = n$ : In this case we have that $E$ is a basis itself. This follows by the maximal linear independence property defining a basis as follows: by *a*) we know that every linearly independent set must have at most $n$ vectors in it. Thus, if $E \subset E'$ and $E'$ is linearly independent, then we must necessarily have $E' = E$, since $E'$ cannot have any more than $n$ vectors. This is just the maximal linear independence property defining a basis. Hence, $E$ is a basis of $V$.

*Case 2:* $k < n$: Now, by *b*), we know that any spanning set of $V$ must have at least $n$ vectors in it. Hence, since $k < n$ we have $\operatorname{span}_{\mathbb{K}} E \subset V$ while $\operatorname{span}_{\mathbb{K}} E \neq V$. We claim that there exists $f_{k+1} \in F$ such that $f_{k+1} \notin \operatorname{span}_{\mathbb{K}} E$. For, if not, then we would have $F \subset \operatorname{span}_{\mathbb{K}} E$, so that $V = \operatorname{span}_{\mathbb{K}} F \subset \operatorname{span}_{\mathbb{K}} E \subset V$, which is absurd as $\operatorname{span}_{\mathbb{K}} E \neq V$. Then, $F_1 = E \cup \{f_{k+1}\}$ is a linearly independent set, by the Elimination Lemma. If $\operatorname{span}_{\mathbb{K}} F_1 = V$ then we have that $F_1$ is a basis and we are done. Otherwise, $\operatorname{span}_{\mathbb{K}} F_1 \neq V$. As before, we can find $f_{k+2} \in F$ such that $f_{k+2} \notin \operatorname{span}_{\mathbb{K}} F_1$ and obtain linearly independent set $F_2 = F_1 \cup \{f_{k+2}\}$. Then, either $\operatorname{span}_{\mathbb{K}} F_2 = V$ and we are done, or $\operatorname{span}_{\mathbb{K}} F_2 \neq V$ and we can define a linearly independent set $F_3$. Proceeding in this manner we either obtain a basis $F_i$, for some $i < n - k$, or we obtain a linearly independent set $F_{n-k}$ and we are back in Case 1, so that $F_{n-k}$ must be a basis. In either case, we find a basis of the required form. □

**Corollary 1.5.16.** *Let* $V$ *be a* $\mathbb{K}$*-vector space such that* $\dim_{\mathbb{K}} V = n$ *and* $E \subset V$*.*

- *If* $E$ *is linearly independent and* $|E| = n$*, then* $E$ *is a basis of* $V$*.*
- *If* $\operatorname{span}_{\mathbb{K}} E = V$ *and* $|E| = n$*, then* $E$ *is a basis of* $V$*.*

*Proof:* The first statement was shown in $c$). The second statement is left to the reader. $\square$

**Corollary 1.5.17.** *Let $V$ be a $\mathbb{K}$-vector space such that $\dim_{\mathbb{K}} V = n$, $U \subset V$ a subspace. Then, $\dim_{\mathbb{K}} U \leq n$. Moreover, if $\dim_{\mathbb{K}} U = n$, then $U = V$.*

*Proof:* Let $\mathcal{B}' \subset V$ be a basis of $U$. Then, $\mathcal{B}'$ is a linearly independent subset of $U$, therefore a linearly independent subset of $V$. Hence, by Proposition 1.5.15, we have that $\mathcal{B}'$ contains no more than $n$ vectors. By the definition of dimension the result follows.

Moreover, suppose that $\dim_{\mathbb{K}} U = n$. Then, there is a subset $\mathcal{B}'$ of $U$ that is linearly independent and contains exactly $n$ vectors. Hence, by the previous Corollary, $\mathcal{B}'$ is a basis of $V$. So, since $\operatorname{span}_{\mathbb{K}} \mathcal{B}' = U$ and $\operatorname{span}_{\mathbb{K}} \mathcal{B}' = V$ we have $U = V$. $\square$

**Corollary 1.5.18.** *Let $V$ be a $\mathbb{K}$-vector space, $U \subset V$ a subspace. Then, any basis of $U$ can be extended to a basis of $V$.*

*Proof:* Let $\mathcal{B}' = \{b_1', \dots, b_r'\}$ be a basis of $U$ and $\mathcal{B} = \{b_1, \dots, b_n\}$ a basis of $V$; in particular, $\operatorname{span}_{\mathbb{K}} \mathcal{B} = V$. Then, by Proposition 1.5.15, part $c$), we can extend $\mathcal{B}'$ to a basis of $V$ using vectors from $\mathcal{B}$. $\square$

**Corollary 1.5.19.** *Let $V$ be a $\mathbb{K}$-vector space, $U \subset V$ a subspace. Then, there exists a subspace $W \subset V$ such that $V = U \oplus W$. Moreover, in this case we have*

$$\dim V = \dim U + \dim W,$$

*and if $\mathcal{B}'$ is any basis of $U$ and $\mathcal{B}''$ is any basis of $W$ then $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}''$ is a basis of $V$.*

*Proof:* Let $\mathcal{B}' = \{b_1', \dots, b_r'\}$ be a basis of $U$ and extend to a basis $\mathcal{B} = \{b_1', \dots, b_r', b_{r+1}, \dots, b_n\}$ of $V$, using the previous Corollary. Then, let $W = \operatorname{span}_{\mathbb{K}}\{b_{r+1}, \dots, b_n\}$. Then, since $\mathcal{B}$ is a basis we have that $V = U + W$ (as every vector in $v$ can be expressed as a linear combination of vectors from $\mathcal{B}$). We need to show that $U \cap W = \{0_V\}$. So, let $x \in U \cap W$. Then, we have

$$x = \lambda_1 b_1' + \dots + \lambda_r b_r' \in U,$$

and

$$x = \mu_1 b_{r+1} + \dots + \mu_{n-r} b_n \in W.$$

Hence,

$$\mu_1 b_{r+1} + \dots + \mu_{n-r} b_n = x = \lambda_1 b_1' + \dots + \lambda_r b_r',$$

giving a linear relation

$$\lambda_1 b_1' + \dots + \lambda_r b_r' - (\mu_1 b_{r+1} + \dots + \mu_{n-r} b_n) = 0_V.$$

Thus, as $\mathcal{B}$ is linearly independent then this linear relation must be trivial so that

$$\mu_1 = \dots = \mu_{n-r} = \lambda_1 = \dots = \lambda_r = 0;$$

hence, $x = 0_V$ so that $U \cap W = \{0_V\}$.

The statement concerning the dimension of $V$ follows from the above proof.

The final statement follows from a dimension count and a simple argument showing that $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}''$ is a linearly independent set. Now we can use Corollary 1.5.16 to deduce that $\mathcal{B}$ is a basis of $V$. The details are left to the reader. $\square$

We end this section with an important formula relating the dimension of subspaces, the so-called *dimension formula*.

**Proposition 1.5.20** (Dimension formula)**.** *Let $V$ be a $\mathbb{K}$-vector space, $U, W \subset V$ two subspaces of $V$. Then,*

$$\dim(U + W) = \dim U + \dim W - \dim U \cap W.$$

*Note that here we are not assuming that $V = U + W$.*

*Proof:* Let $X = U + W$ so that $X \subset V$ is a subspace of $V$ and can be considered as a $\mathbb{K}$-vector space in its own right. Moreover, we have that $U, W, U \cap W \subset X$ are all subspaces of $X$ and $U \cap W$ is a subspace of both $U$ and $W$.

Now, if $U \subset W$ (resp. $W \subset U$) then we have $U + W = W$ (resp. $U + W = U$) and $U \cap W = U$ (resp. $U \cap W = W$). So, in this case the result follows easily.

Therefore, we will assume that $U \not\subseteq W$ and $W \not\subseteq U$ so that $U \cap W \subset U$ and $U \cap W \subset W$ while $U \cap W \neq U, W$. Using the previous Corollary we have that there are subspaces $U' \subset U$ and $W' \subset W$ such that

$$U = (U \cap W) \oplus U', \quad \text{and} \quad W = (U \cap W) \oplus W'.$$

Let $\mathcal{B}_1$ be a basis of $U \cap W$, $\mathcal{B}_2$ a basis of $U'$ and $\mathcal{B}_3$ a basis of $W'$. We claim that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ is a basis of $U + W$. Indeed, since $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of $U$ and $\mathcal{B}_1 \cup \mathcal{B}_3$ is a basis of $W$ (by the previous Corollary), we certainly have that $\text{span}_{\mathbb{K}} \mathcal{B} = U + W$[28]. Furthermore, it is straightforward to show that $\mathcal{B}$ is linearly independent[29] thereby giving that $\mathcal{B}$ is a basis of $U + W$. Thus,

$$\dim(U + W) = \dim U' + \dim U \cap W + \dim W',$$

and

$$\dim U = \dim U' + \dim U \cap W, \quad \text{and} \quad \dim W = \dim W' + \dim U \cap W.$$

Comparing these equations gives the result. □

**Example 1.5.21.** 1. The subset

$$E = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{Q}^3,$$

defines a basis of $\mathbb{Q}^3$. Since $E$ has consists of 3 vectors and $\dim_{\mathbb{Q}} \mathbb{Q}^3 = 3$, we need only show that $E$ is linearly independent (Corollary 1.5.16). So, by Example 1.3.6, this amounts to showing that the homogeneous matrix equation

$$A\underline{x} = \underline{0},$$

has only one solution, namely the zero solution, where $A$ is the matrix whose columns are the vectors in $E$. Now, since we can row-reduce

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim I_3,$$

we find that $E$ is indeed linearly independent, so that it must be a basis, by Corollary 1.5.16.

2. Consider the subset

$$E = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \subset Mat_2(\mathbb{R}).$$

Then, $E$ is a basis of $Mat_2(\mathbb{Q})$. Again we use Corollary 1.5.16: since $E$ has 4 vectors and $\dim_{\mathbb{R}} Mat_2(\mathbb{R}) = 2.2 = 4$ we need only show that $E$ is linearly independent or that it spans $Mat_2(\mathbb{R})$. We will show that $\text{span}_{\mathbb{R}} E = Mat_2(\mathbb{R})$. So, let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

[28] The reader should check this.
[29] Again, this is an exercise left to the reader.

Then, we have

$$A = a_{11}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + a_{22}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \frac{(a_{12}+a_{21})}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + \frac{(a_{12}-a_{21})}{2}\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix},$$

so that $A \in \text{span}_{\mathbb{R}} E$. Since $A$ was arbitrary we must have $\text{span}_{\mathbb{R}} E = Mat_2(\mathbb{R})$.

Furthermore, if we consider the ordered basis

$$\mathcal{B} = \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\right),$$

then the $\mathcal{B}$-coordinate morphism is the linear morphism

$$[-]_{\mathcal{B}} : Mat_2(\mathbb{R}) \to \mathbb{R}^4 \ ; \ A = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix} \mapsto \begin{bmatrix}a_{11}\\(a_{12}+a_{21})/2\\a_{22}\\(a_{12}-a_{21})/2\end{bmatrix}$$

## 1.6 Coordinates

([1], Ch. 5)

**Throughout this section we assume that all $\mathbb{K}$-vector spaces are finite dimensional.**

### 1.6.1 Solving problems

The results of the previous section form the theoretical underpinning of how we hope to solve linear algebra problems in practice. The existence of an ordered basis $\mathcal{B} = (b_1, \dots, b_n)$ of a $\mathbb{K}$-vector space $V$ from Theorem 1.5.10, such that $n = \dim V$, along with Corollary 1.5.6 and Corollary 1.5.7 allow us to introduce the notion of $\mathcal{B}$-coordinates on $V$: we have an isomorphism

$$[-]_{\mathcal{B}} : V \to \mathbb{K}^n \ ; \ v \mapsto [v]_{\mathcal{B}} = \begin{bmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{bmatrix},$$

where $v = \lambda_1 b_1 + \dots + \lambda_n b_n$ is the unique expression determined in Corollary 1.5.6. Then, using Proposition 1.4.12, we know that questions concerning linear independence and spans of subsets in $V$ have the same answers if we translate them to questions in $\mathbb{K}^n$ via the $\mathcal{B}$-coordinate map. Since we are then talking about sets of column vectors we can use row-reduction methods to answer the question that was originally posed concerning vectors in $V$.

So, we have the following approach to solving questions about linear independence/spans of subsets $E \subset V$ in finite dimensional $\mathbb{K}$-vector spaces $V$ (we suppose that $n = \dim V$):

0. If $|E| > n$ then $E$ is linearly dependent; if $|E| < n$ then it is not possible that $E$ spans $V$. This follows from Proposition 1.5.15.
1. Determine an ordered basis $\mathcal{B}$ of $V$ using, for example, Corollary 1.5.16.
2. Using the $\mathcal{B}$-coordinate morphism $[-]_{\mathcal{B}} : V \to \mathbb{K}^n$, determine the set $[E]_{\mathcal{B}} = \{[e]_{\mathcal{B}} \mid e \in E\}$.
3. Using row-reduction determine the linear independence/spanning properties of the set $[E]_{\mathcal{B}}$.
4. By Proposition 1.4.12, linear independence/spanning properties of $[E]_{\mathcal{B}}$ are the same as those of $E \subset V$.

### 1.6.2 Change of basis/change of coordinates

We have just seen an approach to solving linear independence/spanning property problems for a (finite dimensional) $\mathbb{K}$-vector space $V$. However, it is not necessarily true that everyone will choose the same ordered basis $\mathcal{B}$ of $V$: for example, we could choose a different ordering on the same set $\mathcal{B}$, leading to a different ordered basis; or, you could choose an ordered basis that is a completely distinct set from an ordered basis I may choose.

Of course, this should not be a problem when we solve problems as the linear independence/spanning properties of a subset $E$ should not depend on how we want to 'view' that subset, ie, what coordinates we choose. However, given two distinct ordered bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of $V$, it will be the case in general that $[E]_{\mathcal{B}_1}$ and $[E]_{\mathcal{B}_2}$ are different sets so that if we wanted to compare our work with another mathematician we we would need to know how to translate between our two different 'viewpoints' we've adopted, ie, we need to know how to change coordinates.

**Proposition 1.6.1** (Change of coordinates)**.** *Let $\mathcal{B} = \{b_1, \dots, b_n\}$ and $\mathcal{C} = \{c_1, \dots, c_n\}$ be two ordered bases of $V$. Let $P_{\mathcal{C}\leftarrow\mathcal{B}}$ be the $n \times n$ matrix*

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[b_1]_{\mathcal{C}}[b_2]_{\mathcal{C}} \cdots [b_n]_{\mathcal{C}}],$$

*so that the $i^{th}$ column is $[b_i]_{\mathcal{C}}$, the $\mathcal{C}$-coordinates of $b_i$. Then, for every $v \in V$, we have*

$$[v]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[v]_{\mathcal{B}}.$$

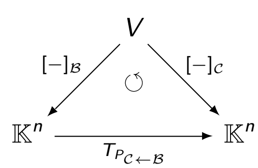
*Moreover, if $A \in Mat_n(\mathbb{K})$ is such that*

$$[v]_{\mathcal{C}} = A[v]_{\mathcal{B}},\ \forall v \in V,$$

*then $A = P_{\mathcal{C}\leftarrow\mathcal{B}}$.*

We call $P_{\mathcal{C}\leftarrow\mathcal{B}}$ the *change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$*. The formula just given tells us that, given the $\mathcal{B}$-coordinates of a vector $v \in V$, to obtain the $\mathcal{C}$-coordinates of $v$ we must multiply the $\mathcal{B}$-coordinate vector of $v$ on the left by $P_{\mathcal{C}\leftarrow\mathcal{B}}$. Moreover, we see that the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$ is uniquely characterised by this property.

**Remark 1.6.2.** 1. We can organise this data into a diagram




where

$$T_{P_{\mathcal{C}\leftarrow\mathcal{B}}} : \mathbb{K}^n \to \mathbb{K}^n\ ;\ \underline{x} \mapsto P_{\mathcal{C}\leftarrow\mathcal{B}}\underline{x}.$$

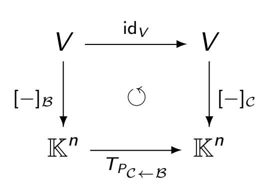
is the linear morphism defined by the matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

The symbol '$\circlearrowleft$' that appears is to be translated as

'the composite morphism $T_{P_{\mathcal{C}\leftarrow\mathcal{B}}} \circ [-]_{\mathcal{B}} : V \to \mathbb{K}^n$ equals the morphism $[-]_{\mathcal{C}} : V \to \mathbb{K}^n$.'

That is, if we start at (the domain) $V$ and follow the arrows either to the left or right then we get the same answer in (the codomain) $\mathbb{K}^n$ (at the bottom right of the diagram). In this case, we say that the **diagram commutes**.

We could also write this diagram as



where $\text{id}_V : V \to V$ is the identity morphism from Example 1.4.8. The reason we are also considering this diagram will become apparent in the following sections.

2. Suppose that $P_{\mathcal{B}\leftarrow\mathcal{C}}$ is the change of coordinate matrix from $\mathcal{C}$ to $\mathcal{B}$. This means that for every $v \in V$ we have

$$[v]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[v]_{\mathcal{C}}.$$

Then, if we want to change back to $\mathcal{C}$-coordinates, we simply multply on the left by $P_{\mathcal{C}\leftarrow\mathcal{B}}$ so that

$$[v]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[v]_{\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}}[v]_{\mathcal{C}}, \ \forall v \in V.$$

This means that the morphism

$$T_{P_{\mathcal{C}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}}} : \mathbb{K}^n \to \mathbb{K}^n \ ; \ \underline{x} \mapsto P_{\mathcal{C}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}}\underline{x},$$

is the identity morphism $\text{id}_{\mathbb{K}^n}$ of $\mathbb{K}^n$; this uses the fact that $V$ is isomorphic to $\mathbb{K}^n$.[30]

We will see later on that this implies that $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{C}}$ are invertible matrices and are inverse to each other:

$$P_{\mathcal{C}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}} = I_n = P_{\mathcal{B}\leftarrow\mathcal{C}}P_{\mathcal{C}\leftarrow\mathcal{B}},$$

where $I_n$ is the $n \times n$ identity matrix.

This should not be surprising: all we have shown here is that the operations 'change coordinates from $\mathcal{B}$ to $\mathcal{C}$' and 'change coordinates from $\mathcal{C}$ to $\mathcal{B}$' are inverse to each other.

Of course, you can also obtain this result knowing that a matrix with linearly independent columns is invertible; this should be familiar to you from your first linear algebra course. However, we have just stated a stronger result: not only have we determined that a change of coordinate matrix is invertible, we have provided what the inverse actually is.

**Example 1.6.3.** 1. Consider the two ordered bases $\mathcal{S}^{(3)} = (e_1, e_2, e_3)$ and

$$\mathcal{B} = \left( \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

of $\mathbb{Q}^3$. Then, what is the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{S}^{(3)}$? We use the formula given above: we have

$$P_{\mathcal{S}^{(3)}\leftarrow\mathcal{B}} = [[b_1]_{\mathcal{S}^{(3)}}[b_2]_{\mathcal{S}^{(3)}}[b_3]_{\mathcal{S}^{(3)}}] = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Therefore, the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{S}^{(3)}$ is simply the matrix whose $i^{th}$ column is the $i^{th}$ basis vector of the ordered basis $\mathcal{B}$.

Moreover, if we want to determine the change of coordinate matrix from $\mathcal{S}^{(3)}$ to $\mathcal{B}$ we need to determine the inverse matrix of $P_{\mathcal{S}^{(3)}\leftarrow\mathcal{B}}$, using row-reduction methods, for example.

2. In general, if $\mathcal{S}^{(n)} = (e_1, \dots, e_n)$ is the standard ordered basis of $\mathbb{K}^n$ and $\mathcal{B} = (b_1, \dots, b_n)$ is any other ordered basis of $\mathbb{K}^n$, then the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{S}^{(n)}$ is

$$P_{\mathcal{S}^{(n)}\leftarrow\mathcal{B}} = [b_1 \ b_2 \ \cdots \ b_n].$$

Again, if we wish to determine the change of coordinate matrix from $\mathcal{S}^{(n)}$ to $\mathcal{B}$ we need to determine the inverse matrix of $P_{\mathcal{S}^{(n)}\leftarrow\mathcal{B}}$. This may not be so easy for large matrices.

[30] Why is this true?