Math 110, Summer 2012 : JCF review problems

Polynomials, representations

1. Determine the minimal polynomials of the following matrices:

- $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$,

- $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$,

- $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$,

- $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

Solution:

- We have
$$\chi_A(t) = (2-t)^2(5-t),$$
and μ_A has the exact same roots as χ_A. Hence, we must have
$$\mu_A = \begin{cases} -\chi_A, \\ (t-2)(t-5). \end{cases}$$
Since
$$(A - 2I_3)(A - 5I_3) = 0_3,$$
we have $\mu_A = (t-2)(t-5)$.

- We have
$$\chi_A(t) = (1-t)(t-(1+\sqrt{-2}))(t-(1-\sqrt{-2})).$$
Hence, we must have
$$\mu_A = -\chi_A.$$

- We have
$$\chi_A(t) = (1-t)^2(3-t),$$
so that
$$\mu_A = \begin{cases} (1-t)(3-t), \\ -\chi_A. \end{cases}$$
Since
$$(I_3 - A)(3I_3 - A) \neq 0_3,$$
then we must have
$$\mu_A = -\chi_A.$$

- We have

$$\chi_A = (1-t)^3(-1-t),$$

so that

$$\mu_A = \begin{cases} (t-1)(t+1), \\ (t-1)^2(t+1), \\ \chi_A. \end{cases}$$

Since

$$(A - I_2)(A + I_3) \neq 0_4, (A - I_2)^2(A + I_2) \neq 0_4,$$

we must have

$$\mu_A = \chi_A.$$

Canonical form of an endomorphism

Determine the Jordan canonical form J of the above matrices. Find $P \in GL_n(\mathbb{C})$ such that $P^{-1}AP = J$.

Solution:

- The JCF is

$$\begin{bmatrix} 2 & & \\ & 2 & \\ & & 5 \end{bmatrix},$$

since μ_A is a product of distinct linear factors, therefore A is diagonalisable.

- The JCF is

$$\begin{bmatrix} 1 & & \\ & 1-\sqrt{-2} & \\ & & 1+\sqrt{-2} \end{bmatrix},$$

since A is diagonalisable (μ_A is a product of distinct linear factors).

- We have the JCF is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

You can determine this following a similar approach as the solution to SH8, Q1.

- The JCF is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

You can determine this in a similar approach as the solution to SH8, Q1.

To determine P you must follow the same approach as for the solution to SH8, Q1, or Practice Exam 2, Q1.

Jordan canonical form

Suppose that you are given a matrix $A \in Mat_5(\mathbb{C})$ such that μ_A is one of the below polynomials. What are the possibilities for the Jordan canonical form of A?

- $\mu_A = (t+1)^2(t-2)^2$,
- $\mu_A = t(t+4)(t-2)$,
- $\mu_A = t^2(t-2)$,
- $\mu_A = (t-5)(t-1)(t+3)^3$.

(*Hint: recall that μ_A is an annihilating polynomial and we can use the Primary Decomposition Theorem to determine a direct sum decomposition of $\mathbb{C}^5$. What are the allowed blocks of the Jordan form of A?*)

Solution:

- The JCF can't be a diagonal, as μ_A is not a product of distinct linear factors. We can't have the JCF of the form

$$\begin{bmatrix} A(-1) & 0 \\ 0 & 2I_k \end{bmatrix}, \text{ or } \begin{bmatrix} A(2) & 0 \\ 0 & -I_l \end{bmatrix},$$

  as then the minimal polynomial would be of the form $(t+1)^2(t-2)$ or $(t+1)(t-2)^2$. Here, $A(-1)$ and $A(2)$ are the -1 and 2 parts of the JCF. Hence, the JCF must be of the form

$$\begin{bmatrix} J(\alpha,1) & & \\ & J(\alpha,2) & \\ & & J(\beta,2) \end{bmatrix}, \text{ or } \begin{bmatrix} J(\alpha,3) & \\ & J(\beta,2) \end{bmatrix},$$

  where $\alpha, \beta \in \{-1, 2\}$, $\alpha \neq \beta$, and $J(\alpha, i)$ is the $i \times i$ α-Jordan block.

- The JCF is a diagonal matrix with entries $0, -4, 2$, and where each possibility appears at least once, and at most three times.

- The JCF is of the form

$$\begin{bmatrix} C(0) & \\ & 2I_k \end{bmatrix},$$

  where $C(0)$ is of the form

$$J(0,4), \text{ or } \begin{bmatrix} J(0,3) & \\ & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} J(0,2) & \\ & J(0,2) \end{bmatrix}, \text{ or } \begin{bmatrix} J(0,2) & & \\ & 0 & \\ & & 0 \end{bmatrix}, \text{ or }$$

$$J(0,3), \text{ or } \begin{bmatrix} J(0,2) & \\ & 0 \end{bmatrix}, \text{ or } J(0,2).$$

  where $J(0, i)$ is the $i \times i$ 0-Jordan block.

- The JCF must take the form

$$\begin{bmatrix} 5 & & \\ & 1 & \\ & & J(-3,3) \end{bmatrix}.$$