HERMANN GRASSMANN AND THE CREATION OF LINEAR ALGEBRA

DESMOND FEARNLEY-SANDER

There is a way to advance algebra as far beyond what Vieta and Descartes have left us as Vieta and Descartes carried it beyond the ancients. ...We need an analysis which is distinctly geometrical or linear, and which will express *situation* directly as algebra expresses *magnitude* directly.

—Leibniz [**16**, p. 382]

1. Introduction. From Pythagoras to the mid-nineteenth century, the fundamental problem of geometry was to relate numbers to geometry. It played a key role in the creation of field theory (via the classic construction problems), and, quite differently, in the creation of linear algebra. To resolve the problem, it was necessary to have the modern concept of real number; this was essentially achieved by Simon Stevin, around 1600, and was thoroughly assimilated into mathematics in the following two centuries. The integration of real numbers into geometry began with Descartes and Fermat in the 1630's, and achieved an interim success at the end of the eighteenth century with the introduction into the mathematics curriculum of the traditional course in analytic geometry. From the point of view of analysis, with its focus on functions, this was entirely satisfactory; but from the point of view of geometry, it was not: the method of attaching numbers to geometric entities is too clumsy, the choice of origin and axes irrelevant and (in view of Euclid) unnecessary. Leibniz, in 1679, had mused upon the possibility of a universal algebra, an algebra with which one would deal directly and simply with geometric entities. The possibility is already suggested by a perusal of Euclid. For example, if D is a point in the side BC of a triangle ABC, then

$$\frac{BD}{BC}=\frac{ABD}{ABC};$$

this ancient theorem begs to be proved by simply multiplying numerator and denominator on the left by A. The geometric algebra of which Leibniz dreamed, and in which the concept of real number is thoroughly assimilated, was created by Hermann Grassmann in the mid-nineteenth century.

Grassmann looked on geometry as it might well be considered today but is not, as being applied mathematics: In his view, there is a part of mathematics, linear algebra, that is applicable to a part of the physical world, chalk figures on a blackboard or objects in space; and geometry, as the business of relating the two, does not belong to mathematics pure and simple. One may say without great exaggeration that Grassmann invented linear algebra and, with none at all, that he showed how properly to apply it in geometry. Linear algebra has become a part of the mainstream of mathematics, though Grassmann gets scant credit for it; but its application to geometry, affine and Euclidean, is remembered only in a half-baked version in which the notion of vector is all important and the notion of point is unnecessary. The reason is that there are other applications of linear algebra which are of greater practical importance (though they are no more interesting) than geometry—namely, within mathematics, to function spaces, and, within physics, to forces and other vectorial entities. [And yet, oddly enough, Grassmann's geometry *is* better suited to physics, since, for example, it distinguishes the notions of (polar) vector and axial vector; modern physics texts (such as Feynman [9]), in attempting to explain

Desmond Fearnley-Sander's M.Sc. thesis at the Australian National University was supervised by R. E. Edwards. After two years postgraduate study at the University of Washington he left, undoctored, to take up an assistant-professorship at the California State College at San Bernardino. He lectured at the University of Western Australia for six years and took up his present post at the University of Tasmania in 1975. He has spent periods of sabbatical leave at Trinity College, Dublin, and at Edinburgh University. His main mathematical interests are linear algebra and its history, and the history of analysis, 1870–1930, and he is at present writing a history of linear algebra.

the physical distinction between the two types of entity, are handicapped by the fact that in the accepted model both have the same mathematical representation.]

2. Grassmann's life. A biography of Grassmann, by Friedrich Engel, may be found in the *Collected Works* [**13**, III.2] and also, more briefly, in Michael Crowe's scholarly work [**6**], the main modern source in English of which I am aware.

Hermann Gunther Grassmann was born in Stettin in 1809, lived there most of his life, and died in 1877. He was one of twelve children and, after marrying at the age of 40, fell short of his father in himself siring only eleven. He spent three years in Berlin studying theology and philology. He had no university mathematical training, nor did he ever hold a university post, though he repeatedly sought one. His life was spent as a schoolteacher.

His major mathematical works are *A Theory of Tides*, which is a kind of thesis written in 1840 in the hope of improving his status as a teacher and which was unpublished until the appearance of the *Collected Works* between 1894 and 1911; a book known briefly as the *Ausdehnungslehre* (literally "Theory of Extension"), which was published in 1844 and almost totally ignored, though it was drawn to the attention of Möbius, Gauss, Kummer, Cauchy, and others; and the *Ausdehnungslehre* of 1862, which was a new work on the same subject, rather than a new edition, and which met an equally cold reception. (Both are included in [**13**].) His many papers include important contributions to physics as well as to mathematics. He also wrote textbooks in mathematics and languages, edited a political journal for a time, and produced a translation of the *Rig Veda* and a huge commentary on it which, according to *Encyclopaedia Britannica*, is still used today. For his work in philology he received, in the last year of his life, an honorary doctorate. His achievements in mathematics were virtually unrecognized, and it has taken a century for their importance to become clearly visible.

While I intend to devote my attention to his linear algebra and geometry, there are two other contributions of Grassmann to mathematics which may be mentioned. In an arithmetic text [**12**] published in 1861, he defined the arithmetic operations for integers inductively and he proved their properties—commutativity, associativity, distributivity. He thus anticipated in its most important aspects Peano's treatment [**19**] of the natural numbers, published 28 years later. Peano generously acknowledges this, but in the naming game by which History distributes fame to the creators of mathematics Peano is a winner, Grassmann a loser. Dedekind, who published a similar development [**7**] of the natural numbers in 1888, makes no mention of Grassmann.

A feature of Grassmann's work, far in advance of the times, is the tendency toward the use of implicit definition—in which a mathematical entity is characterized by means of its formal properties rather than being obtained by an explicit construction. For example, in the *Ausdehnungslehre* of 1844 he comes very close indeed to the abstract notion of a (not necessarily associative) ring; what is lacking is the language of set theory. This is the second contribution I wanted to mention. Incidentally, the first formal definition of a ring was given by Fraenkel [**10**] in 1915.

3. The invention of linear algebra. From the beginning, Grassmann distinguished linear algebra, as a formal theory independent of any interpretation, from its application in geometry. However, in the first *Ausdehnungslehre*, the algebra is intermixed with its geometric interpretation—indicating, very interestingly, how he came upon the ideas. Those who did read his work in the late nineteenth century found it easier to follow the 1862 *Ausdehnungslehre*, in which, in modern style, the full development of the mathematical theory precedes its application, and in outlining his linear algebra I shall mainly follow the latter work; it should be borne in mind, though, that some of the ideas have their origin as much as two decades previously.

The definition of a linear space (or vector space) came into mathematics, in the sense of becoming widely known, around 1920, when Hermann Weyl [**22**] and others published formal definitions. In fact, such a definition had been given thirty years previously by Peano [**18**], who was thoroughly acquainted with Grassmann's mathematical work. Grassmann did not put down a formal definition—again, the language was not available—but there is no doubt that he had the concept. Beginning with a collection of "units" $e_1, e_2, e_3, \ldots$, he effectively defines the free linear space which they generate; that is to say, he considers formal linear combinations $\Sigma \alpha_i e_i$, where the α_i are real numbers, defines addition and multiplication by real numbers by setting

$$\Sigma \alpha_i e_i + \Sigma \beta_i e_i = \Sigma(\alpha_i + \beta_i) e_i$$

and

$$\alpha(\Sigma \alpha_i e_i) = \Sigma(\alpha \alpha_i) e_i,$$

and formally proves the linear space properties for these operations. (At the outset, it is not clear whether the set of units is allowed to be infinite, but finiteness is implicitly assumed in some of his proofs.) He then develops the theory of linear independence in a way which is astonishingly similar to the presentation one finds in modern linear algebra texts.

He defines the notions of subspace, independence, span, dimension, join and meet of subspaces, and projections of elements onto subspaces. He is aware of the need to prove invariance of dimension under change of basis, and does so. He proves the Steinitz Exchange Theorem, named for the man who published it **[20]** in 1913 (and who, incidentally, defined a linear space in terms of "units" in the same way Grassmann did). Among other such results, he shows that any finite set has an independent subset with the same span and that any independent set extends to a basis, and he proves the important identity

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

He obtains the formula for change of coordinates under change of basis, defines elementary transformations of bases, and shows that every change of basis (equivalently, in modern terms, every invertible linear transformation) is a product of elementaries.

4. Products. In a paper **[11]** published in 1855, Grassmann defines a product of elements of a linear space by setting

$$(\Sigma \alpha_i e_i)(\Sigma \beta_j e_j) = \Sigma \alpha_i \beta_j e_i e_j,$$

and he proves distributivity. (In this paper the scalars are explicitly allowed to be complex.) If the $e_i e_j$ are themselves linear combinations of the e_i, we have here the concept of an algebra. Instead of following this path (though he did later observe that the algebra of quaternions is a special case), Grassmann singles out particular products by "equations of condition"

$$\Sigma \xi_{ij} e_i e_j = 0,$$

and, observing as a disadvantage of this notion that it lacks invariance under change of basis, he proceeds to characterize those products whose conditioning equations are invariant under various substitutions.

Grassmann's declared motive for publishing this paper was to claim priority for some results that had been published by Cauchy. The interesting story is related by Engel. In 1847 Grassmann had wanted to send a copy of the *Ausdehnungslehre* to Saint-Venant (to show that he had anticipated some of Saint-Venant's ideas on vector addition and multiplication), but, not knowing the address, Grassmann sent the book to Cauchy, with a request to forward it. Cauchy never did so. And six years later Cauchy's paper **[4]** appeared in *Comptes Rendus*. Grassmann's comment was that, on reading this, "I recalled at a glance that the principles which are there established and the results which are proved were exactly the same as those which I published in 1844, and of which I gave at the same time numerous applications to algebraic analysis, geometry, mechanics and other branches of physics." An investigating committee of three members of the French Academy, including Cauchy himself, never came to a decision on the question of priority.

In the two *Ausdehnungslehren*, Grassmann singles out for special attention those products, which he calls *linear products*, for which the conditioning equations are invariant under change of basis. He shows that, apart from two trivial cases, there are only two possible types of linear product: it must be the case that

$$\text{either (1) } e_ie_j = e_je_i$$

$$\text{or (2) } e_ie_j = -e_je_i \text{ (and, in particular, } e_i^2 = 0)$$

for all i and j. Whereas in the 1855 paper he does not consider higher order products, in the *Ausdehnungslehren* he examines in detail products of the second type, extended, by imposing associativity, to allow multiplication of products of the original simple units (such a product being called a *compound unit*) and, by imposing distributivity, to allow multiplication of linear combinations of compound units (such a combination being called a *form*). Condition (2) entails certain relations among compound units of higher order (for example that $e_1e_3e_2 = -e_1e_2e_3$), but it is assumed that no other relations hold. Dimension plays a role here, since, for example, in the three-dimensional case we have only a single independent third-order unit, say $e_1e_2e_3$, and this forces independence of the three units e_1e_2, e_2e_3, and e_3e_1 of order 2, because

$$\xi_1e_1e_2 + \xi_2e_2e_3 + \xi_3e_3e_1 = 0$$

implies, when we multiply by e_3, that $\xi_1e_1e_2e_3 = 0$ and hence $\xi_1 = 0$ (and, similarly, $\xi_2 = 0 = \xi_3$).

This multiplication is nowadays called *exterior multiplication*. Half a century later, in treating Grassmann's ideas at length in his *Universal Algebra* [23], A. N. Whitehead explicitly excludes forms of mixed degree like $e_1 + e_1e_2$ on the rather metaphysical ground that they are "meaningless"; he thus admits arbitrary products of linear combinations of simple units, but not arbitrary linear combination of products. Whitehead's objection itself would have been meaningless to Grassmann and, although he never explicitly brings in such forms, neither, so far as I can see, does he explicitly exclude them. If I am right about this, then Grassmann has the full exterior algebra, while Whitehead's presentation (like many modern treatments of exterior products) restricts consideration to a graded linear space (a far less tidy structure than an algebra).

The full development of exterior algebra as Grassmann did it (in particular the essential invariance under change of basis) is complicated and must be omitted. Perhaps the most important fact is that elements $a_1, a_2, \ldots, a_k$ of the original linear space are linearly independent if and only if $a_1a_2\cdots a_k \neq 0$; Grassmann proves this in the modern way and gives the following application (with notation precisely as I have written it) to a system of n linear equations in n unknowns,

$$\alpha_1^{(1)}x_1 + \alpha_2^{(1)}x_2 + \cdots + \alpha_n^{(1)}x_n = \beta^{(1)}$$
$$\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \cdots + \alpha_n^{(2)}x_n = \beta^{(2)}$$
$$\cdots$$
$$\alpha_1^{(n)}x_1 + \alpha_2^{(n)}x_2 + \cdots + \alpha_n^{(n)}x_n = \beta^{(n)}.$$

Introducing (independent) units $e^{(1)}, e^{(2)}, \ldots, e^{(n)}$ and quantities

$$a_1 = \alpha_1^{(1)}e^{(1)} + \alpha_1^{(2)}e^{(2)} + \cdots + \alpha_1^{(n)}e^{(n)}$$
$$a_2 = \alpha_2^{(1)}e^{(1)} + \alpha_2^{(2)}e^{(2)} + \cdots + \alpha_2^{(n)}e^{(n)}$$
$$\cdots$$
$$a_n = \alpha_n^{(1)}e^{(1)} + \alpha_n^{(2)}e^{(2)} + \cdots + \alpha_n^{(n)}e^{(n)}$$

and

$$b = \beta^{(1)}e^{(1)} + \beta^{(2)}e^{(2)} + \cdots + \beta^{(n)}e^{(n)},$$

we have

$$b = x_1a_1 + x_2a_2 + \cdots + x_na_n,$$

and so $ba_2a_3\cdots a_n = x_1a_1a_2\cdots a_n, \ldots$ Thus if $a_1a_2\cdots a_n \neq 0$ we have the unique solution

$$x_1 = \frac{ba_2a_3 \cdots a_n}{a_1a_2a_3 \cdots a_n}, \; x_2 = \ldots, \ldots .$$

(Equivalently, x_1 is the number obtained by dividing the (non-zero) determinant of the matrix $(\alpha_j^{(i)})$ into the determinant of the matrix obtained from $(\alpha_j^{(i)})$ by replacing each $\alpha_1^{(i)}$ by $\beta^{(i)}$.) This is Cramer's rule [5]. The same elegant derivation (but without the double subscript-superscript notation) is given in the first *Ausdehnungslehre*. It is one of the techniques that occurs in the above-mentioned paper of Cauchy.

5. Inner products. Grassmann derives the concept of inner product from that of exterior product in a very interesting way. Working in the algebra generated by the simple units $e_1, e_2, \ldots, e_n$ (subject, as always, to (2)), he defines the *supplement* $|E$ of a compound unit E as $+1$ or -1 times the product of those of the simple units that are not factors of E, the sign $+$ or $-$ being chosen in such a way that

$$E|E = +e_1e_2 \cdots e_n.$$

For example, in the three-dimensional case,

$$|e_1 = e_2e_3 \quad \text{and} \quad |e_1e_3 = -e_2.$$

The supplement is extended to linear combinations of compound units of the same order by linearity:

$$|\Sigma \alpha_j E_j = \Sigma \alpha_j |E_j.$$

(The map $A \to |A$ is nowadays called the *Hodge star-operator*.) If E and F are forms of the same order, then necessarily $E|F$ is a multiple of $e_1e_2 \cdots e_n$; indeed, since the forms of order n make up a one-dimensional linear space, one may identify them with the scalars, and this Grassmann does, setting $e_1e_2 \cdots e_n = 1$. Although $E|F$ makes sense for forms E and F of different order, it is only in the case where both orders are the same (in particular, where they are both 1) that $E|F$ is a number. Noting that $E|F$ is linear in both E and F, Grassmann calls $E|F$ the *inner product* of E and F. Its restriction to the original linear space (the space of forms of order 1) is indeed an inner product in the modern sense, since, as he shows,

$$e_i|e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and hence

$$\Sigma \alpha_i e_i | \Sigma \beta_j e_j = \Sigma \alpha_i \beta_i.$$

He calls $\sqrt{a|a}$ the *numerical value* of a. The notion of a complete orthonormal set is introduced, and it is shown that such a set must be independent and that in theorems involving the inner product the original system of units may be replaced by any such set. The notions of orthogonal complement and orthogonal projection are investigated; the *Gram-Schmidt Process* is at least implicitly involved in this.

6. Linear transformations. For the linear transformation that carries the basis elements $e_1, e_2, \ldots, e_n$ to $b_1, b_2, \ldots, b_n$, respectively, Grassmann writes

$$Q = \frac{b_1, b_2, \ldots, b_n}{e_1, e_2, \ldots, e_n},$$

and considers Q to be a generalized quotient. While there are obvious problems with this notation, it does have a certain elegance; for example, if $b_1, b_2, \ldots, b_n$ are independent then the inverse of Q is

$$\frac{e_1, e_2, \ldots, e_n}{b_1, b_2, \ldots, b_n}.$$

He effectively obtains the matrix representation of Q by showing that it may be written

$$Q=\Sigma\alpha_{r,s}E_{r,s}$$

where

$$E_{r,s}=\frac{0,\ldots,0,e_s,0,\ldots,0}{e_1,\ldots,e_r,\ldots,e_n}.$$

The *determinant* of Q is defined to be the number

$$\frac{b_1b_2\cdots b_n}{e_1e_2\cdots e_n},$$

eigenvalues and *eigenvectors* are introduced (though different terms are used), and the fact that the eigenvalues are roots of the characteristic polynomial is demonstrated as follows. Suppose that

$$Qx=\rho x$$

where $x=\Sigma\xi_ie_i\neq 0$. Writing $c_i=(\rho-Q)e_i$, we see that $\Sigma\xi_ic_i=0$; thus $c_1,c_2,\ldots,c_n$ are dependent and hence

$$[(\rho-Q)e_1][(\rho-Q)e_2]\cdots[(\rho-Q)e_n]=0.$$

Since $e_1e_2\cdots e_n\neq 0$, this is equivalent to the vanishing of the determinant of the linear map $\rho-Q$. It is shown that the eigenvectors corresponding to distinct eigenvalues are independent, and the *spectral theorem* for a symmetric Q is proved. Turning to a general linear transformation Q, Grassmann shows, in effect (by constructing an appropriate basis), that the whole space may be decomposed as the direct sum of invariant subspaces W_ρ where ρ ranges through the characteristic roots of Q, and each W_ρ is the kernel of $(Q-\rho)^k, k$ being the algebraic multiplicity of ρ. While the *spectral theorem* had been proved by Weierstrass **[21]** in 1858 (and, for the case of n distinct eigenvalues, by Cauchy **[3]** in 1829), it appears that Grassmann was the first to prove the latter result, which is sometimes called the *primary decomposition theorem*; it goes part of the way to obtaining Jordan's canonical form, published in 1870 **[15]** (the remaining step being to reduce the nilpotent maps $Q-\rho$: $W_\rho\to W_\rho$).

7. Geometry. In the *Ausdehnungslehre* of 1844, Grassmann describes the geometric considerations which led him to the theory that we now call linear algebra. Musing on the formula

$$AB+BC=AC, \tag{3}$$

which one might find in old geometry texts, to describe a relationship between lengths that holds for collinear points A, B and C, with B between A and C, he realized that the formula remains valid regardless of the order of the three collinear points, *provided that one sets*

$$BA=-AB; \tag{4}$$

for example, if C lies between A and B then (3) follows from the fact that

$$AB=AC+CB=AC-BC.$$

Over many years Grassmann carefully investigated the consequences of (4), which is the special property that defines an exterior algebra. His development of geometry is complicated, and we shall give here an oversimplified presentation that brings one rapidly to the heart of the matter; those who regard it as criminal to attempt a modern paraphrase of old mathematics should read no further. To avoid (merely notational) complications we shall, like Grassmann, restrict ourselves to three dimensions.

Beginning with the basic material of geometry, numbers and points, we permit them to be combined by formal operations of sum and product, assuming the elementary algebraic rules for such operations, but subject to the condition that (4) holds for all points A and B and also that

$$\alpha A=A\alpha \tag{5}$$

for all real numbers α and points A. From (4) we deduce that every point A has $A^2=0$; thus the square of a point is a number, and we explicitly assume that it is not a point:

$$0 \text{ is not a point.} \tag{6}$$

We need a rule by which to interpret geometrically the entities which occur in this formal algebra. For a pair of points A and B, and positive real numbers α and β with $\alpha+\beta=1$, write

$$P=\alpha A+\beta B.$$

Then immediately we have

$$AP=\beta AB, PB=\alpha AB \quad \text{and} \quad AP+PB=AB;$$

these formulas suggest that P should be interpreted as the unique point which divides the line segment from A to B in the ratio β to α. We do so interpret P; it is here, via the bijection $\alpha \to \alpha A+\beta B$ between $[0,1]$ and the line segment, that numbers enter geometry, and *all* other geometric interpretations follow from this one. To give an immediate example, the interpretation of $P=\alpha A+\beta B$ with $\alpha \leqslant 0$ and $\beta>0$ and $\alpha+\beta=1$ is forced by the fact that, equivalently

$$B=\left(-\frac{\alpha}{\beta}\right)A+\left(\frac{1}{\beta}\right)P,$$

where $-\alpha/\beta \geqslant 0, 1/\beta>0$ and $(-\alpha/\beta)+1/\beta=1$. The *line* through A and B is the set of all $\alpha A+\beta B$ with $\alpha+\beta=1$, and accordingly we assume that

$$\text{if } A \text{ and } B \text{ are points and } \alpha+\beta=1 \text{ then } \alpha A+\beta B \text{ is a point.} \tag{7}$$

To sum up, we consider a ring Ω, in which the number 1 is a unit element and which is generated by $R \cup P$, where R is the set of real numbers and P is a set whose elements are called *points*, subject to the conditions (4), (5), (6), and (7); equivalently, Ω may be regarded as an algebra generated by P. Dimensionality comes in with the assumption that

$$\Omega \text{ is generated, as an algebra, by four elements of } P \text{ but not by three,} \tag{8}$$

and, finally, ensuring non-triviality of multiplication, that

$$\text{there exist points } A, B, C, D \text{ with } ABCD \neq 0. \tag{9}$$

Existence and uniqueness of such a structure are proved in [**24**]. We then have a model for the geometry of three-dimensional space; the multiplication is an exterior product and Grassmann's abstract theory may now be brought to bear in a situation where geometric interpretation is possible.

The difference of two points is a *vector*; here the interpretation is forced by the identity

$$B-A=C-D \Leftrightarrow \tfrac{1}{2}(A+C)=\tfrac{1}{2}(B+D).$$

The sum of a vector X and a point A is a point, since

$$A+X=B \Leftrightarrow X=B-A.$$

And a product of a number and a vector is a vector, since for $X=B-A$ we have $\alpha X=P-A$ where $P=(1-\alpha)A+\alpha B$; this also entails that αX is to be interpreted as having the same direction as X and α times its length.

Here is a classic theorem which with just these ideas becomes trivial: if, in a triangle with vertices A, B, and C, the points D and E, respectively, divide the side from A to B and the side from A to C in equal ratios, then the line segment from D to E is parallel to the one from B to C and the ratio of their lengths is the appropriate number. The proof is one line:

$$D=\alpha A+\beta B, E=\alpha A+\beta C \Rightarrow D-E=\beta(B-C).$$

The hypotheses (8) and (9) imply that the space of all fourth order forms is one-dimensional: if A, B, C, and D are independent points then any product $A'B'C'D'$ is a multiple of $ABCD$ and one may show that the ratio must be interpreted as the ratio of the oriented volumes of the associated tetrahedra. In particular, $ABCD=A'B'C'D'$ means that the two tetrahedra have the same orientation and volume. This in turn forces one to interpret $ABC=A'B'C'$ as meaning that the associated triangles are coplanar and have the same orientation and area; and $AB=A'B'$ as meaning that the associated line segments lie on the same line and are equal in length and direction.

We may now prove the converse of the above result about triangles in exactly the way that it was done by Euclid. If the line segments from D to E and from B to C are parallel, then

$$\frac{DB}{AB}=\frac{DBC}{ABC}=\frac{EBC}{ABC}=\frac{EC}{AC},$$

where we have used the fact that for a suitable real α

$$DBC=[E+\alpha(B-C)]BC=EBC.$$

Again, here is the proof that the diagonal of a parallelogram bisects it:

$$D=C+A-B\Rightarrow ABD=-ABC.$$

These examples give an indication, if no more, of the power of Grassmann's geometric interpretation of linear algebra. They are results of affine geometry, but by introducing an inner product one gets equally transparent algebraic proofs of the theorems of Euclidean geometry and trigonometry. (Incidentally, Grassmann, with his inclination toward abstraction and generality, displays little interest in proving known results of geometry, and these proofs are mine; many other examples are given by Forder [8].)

8. Contemporary and later developments. Though in 1844 he had been unaware of their work, Grassmann later acknowledged that in some respects his theory was anticipated, in particular, by the concept of vector addition of Bellavitis [1] and others, and by the barycentric calculus of Möbius [17]. But no one had approached the elegant simplicity of the formula

$$(C-B)+(B-A)=C-A$$

which, for Grassmann, forces the interpretation of the sum of two vectors; and Möbius had not exploited the full possibilities of his notation

$$P=\alpha A+\beta B+\gamma C,$$

which, even today, in his brilliant [2], is dismissed by Boyer as inferior to the homogeneous coordinate representation $P=(\alpha,\beta,\gamma)$. The key to the difference between Möbius and Grassmann is that, whereas for Möbius the equation $\alpha A+\beta B=\alpha' A+\beta' B$ entails merely that $\alpha:\beta=\alpha':\beta'$, for Grassmann it implies that $\alpha=\alpha'$ and $\beta=\beta'$; the one concept is appropriate to projective geometry, the other to Euclidean.

The story of Hamilton's invention of the quaternions [14] in 1843 and of the subsequent influence of Hamilton and of Grassmann in the emergence of vector analysis is well told by Crowe. But vector analysis in Crowe's sense of the term is a subject that has ceased to exist, or should have; it has been absorbed by linear algebra. It would not be easy to estimate the relative influences of the two men in the development of linear algebra as we know it, but there is no doubt that Grassmann in his own work came far closer to it than Hamilton or any of his contemporaries. Even in those cases where forerunners may be discerned, his results, and especially his methods, were highly original. All mathematicians stand, as Newton said he did, on the shoulders of giants, but few have come closer than Hermann Grassmann to creating, single-handedly, a new subject.

9. Conclusion. The genius of Descartes revealed itself in his decision to drop the ancient convention that a product of line segments is a rectangle. In Grassmann's geometry a product of line segments is again a higher-dimensional object. It is a return to Euclid, but to Euclid with a difference, the difference that had been dreamed of by Leibniz. But Grassmann's geometry (as distinct from his linear algebra) has been largely forgotten. Perhaps this is because, at the moment when it might have been remembered, Hilbert, with his immense prestige, closed the book of Euclid by showing that Euclid's program, rigorously carried through, was too tedious for anyone to bother with. Grassmann's way is not tedious; properly done, it is simple, direct, and powerful, and perhaps the book should be opened again.

I conclude with a quotation (as translated in [6, p. 89]) from the preface to the 1862 *Ausdehnungslehre*:

> I remain completely confident that the labour I have expended on the science presented here and which has demanded a significant part of my life as well as the most strenuous application of my powers, will not be lost. It is true that I am aware that the form which I have given the science is imperfect and must be imperfect. But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remain unused for another seventeen years or even longer, without entering into the actual development of science, still that time will come when it will be brought forth from the dust of oblivion and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me (as I have until now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich them further, yet there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into a living communication with contemporary developments. For truth is eternal and divine.

This is an expanded version of a paper presented at the 1979 Summer Research Institute of the Australian Mathematical Society. The author thanks John Fox for pointing out an error in the original manuscript.

References

1. Giusto Bellavitis, Saggio di applicazioni di un nuovo methodo di geometria analytica (Calcolo delle equipollenze), Ann. Sci. Regno Lombardo-Veneto, Padova (1835) 246–247.

2. Carl B. Boyer, History of Analytic Geometry, New York, 1956.

3. Augustin Cauchy, Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planètes, Exercices de Mathématiques, Paris, 1829.

4. ______, Sur les clefs algébriques, Comptes Rendus de l'Acad. Sci., 36 (1853) 70–75 and 129–136.

5. Gabriel Cramer, Introduction à l'analyse des lignes courbes algébriques, Geneva, 1750.

6. Michael Crowe, A History of Vector Analysis, Notre Dame, 1967.

7. Richard Dedekind, Was sind und was sollen die Zahlen, Braunschweig, 1888.

8. Henry George Forder, The Calculus of Extension, Cambridge, 1941.

9. Richard P. Feynman, Robert B. Leighton, and Matthew Sands, Lectures on Physics, Addison-Wesley, 1963.

10. A. A. Fraenkel, Über die Teiler der Null und die Zerlegung von Ringen, J. Reine Angew. Math, 145 (1915) 139–176.

11. Hermann Grassmann, Sur les différents genres de multiplication, Crelle's Journal, 49 (1855) 123–141.

12. ______, Lehrbuch der Arithmetik, Berlin, 1861.

13. ______, Gesammelte Mathematische und Physikalische Werke, 3 volumes in 6 parts, Leipzig, 1894–1911.

14. William Rowan Hamilton, On quaternions or on a new system of imaginaries in algebra, Phil. Mag. 3rd Ser. (1844) 10–13.

15. Camille Jordan, Traité des substitutions et des équations algébriques, Paris, 1870.

16. Gottfried Wilhelm Leibniz, Philosophical Papers and Letters, I, trans. and ed. Leroy E. Loemker, Chicago, 1956.

17. August Ferdinand Möbius, Der barycentrische Calcul, Leipzig, 1827.

18. Giuseppe Peano, Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazioni della logica deduttiva, Torino, 1888.

19. ______, Arithmetices principia, nova methoda exposita, Torino, 1889.

20. E. Steinitz, Bedingt konvergente Reihen und konvexe Systeme, J. Reine Angew. Math., 143 (1913) 128–175.

21. Karl Weierstrass, Über ein die homogenen Funcktionen zweiten Grades betreffendes Theorem, nebst Anwendung desselben auf die Theorie der kleinen Schwingungen, Monatsh Akad. Berlin (1858) 207–220.

22. Hermann Weyl, Raum, Zeit und Materie, Berlin, 1923.

23. Alfred North Whitehead, A Treatise on Universal Algebra with Applications, Cambridge, 1898.

Added in proof:

24. Desmond Fearnley-Sander, Affine geometry and exterior algebra, Houston J. Math. (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TASMANIA, GPO BOX 252C, HOBART 7001, AUSTRALIA.