Math 110, Summer 2012 : Bilinear forms review problems

Basics

1. Determine which of the following bilinear forms are symmetric/antisymmetric/neither, nondegenerate:

- $B_A \in \mathrm{Bil}_{\mathbb{Q}}(\mathbb{Q}^3)$, where $A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}$,

- $B_A \in \mathrm{Bil}_{\mathbb{R}}(\mathbb{R}^4)$, where $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 5 \end{bmatrix}$,

- $B : Mat_2(\mathbb{R}) \times Mat_2(\mathbb{R}) \to \mathbb{R}$; $(A, B) \mapsto \mathrm{tr}(A^t XB)$, where $X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

- $B : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$; $(\underline{u}, \underline{v}) \mapsto u_1v_2 + u_2v_1 + u_1v_4 + u_4v_1 + u_2v_2 + u_2v_4 + u_4v_2 + u_4v_4 + 2u_3v_3$.

- $B : Mat_2(\mathbb{R}) \times Mat_2(\mathbb{R}) \to \mathbb{R}$; $(A, B) \mapsto \mathrm{tr}(A^t XB)$, where $X = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

Canonical forms of symmetric nondegenerate real bilinear forms

Determine the canonical form of the following real symmetric nondegenerate bilinear forms.

- $B_A \in \mathrm{Bil}_{\mathbb{R}}(\mathbb{R}^3)$, where $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & -1 \end{bmatrix}$,

- $B_A \in \mathrm{Bil}_{\mathbb{R}}(\mathbb{R}^3)$, where $A = \begin{bmatrix} -3 & 2 & -2 \\ 2 & 1 & 0 \\ -2 & 0 & 5 \end{bmatrix}$,

- $B_A \in \mathrm{Bil}_{\mathbb{R}}(\mathbb{R}^4)$, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & -3 & 1 \\ 2 & -3 & 1 & 0 \\ -1 & 1 & 0 & 5 \end{bmatrix}$.

Which of the bilinear forms are inner products? For those that are not inner products determine $\underline{x}$ such that $B_A(\underline{x}, \underline{x}) < 0$.

Solution:

- We have
$$\underline{x}^t A \underline{x} = (x_1 - x_2)^2 + 2(x_2 + x_3)^2 - 3x_3^2,$$
so if
$$P = Q^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{3} \end{bmatrix}^{-1},$$

then

$$P^t A P = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}.$$

This is NOT an inner product. We have

$$B_A\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = -3 < 0.$$

- We have

$$\underline{x}^t A \underline{x} = -3(x_1 - \frac{2}{3}(x_2 - x_3))^2 + \frac{7}{3}(x_2 - \frac{4}{7}x_3)^2 + \frac{133}{21}x_3^2.$$

If we set

$$P = Q^{-1} = \begin{bmatrix} \sqrt{3} & -\frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \sqrt{\frac{7}{3}} & \frac{4}{\sqrt{21}} \\ 0 & 0 & \sqrt{\frac{133}{21}} \end{bmatrix}^{-1},$$

then

$$P^t A P = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix}.$$

This is NOT an inner product. We have

$$B_A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = -3 < 0.$$

Projections, Gram-Schmidt

Determine $\text{proj}_U v$, for the given subspace $U \subset V$ and $v \in V$. You will need to determine an orthogonal basis of U using Gram-Schmidt (with respect to the 'dot product').

- $U = \ker T_A$, where $A = \begin{bmatrix} 1 & -3 & 1 \\ 3 & -1 & 0 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

- $U = \text{im}\, T_A$, where $A = \begin{bmatrix} -1 & 0 & 2 & -1 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$,

- $U = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$, $v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Determine $U^\perp$ for the above subspaces U.

Solution:

- We have that
$$\begin{bmatrix}1 & -3 & 1\\ 3 & -1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1/8\\ 0 & 1 & -3/8\end{bmatrix},$$
so that
$$U = \operatorname{span}_{\mathbb{R}} \left\{ \begin{bmatrix}1\\3\\8\end{bmatrix} \right\}.$$
Hence,
$$\left(\begin{bmatrix}1\\3\\8\end{bmatrix} \right),$$
is an orthogonal basis of U.

- We have
$$\operatorname{im} T_A = \operatorname{span}_{\mathbb{R}} \left\{ \begin{bmatrix}-1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\2\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix} \right\},$$
and since
$$\begin{bmatrix}-1 & 2 & -1\\ 2 & 2 & 1\\ 0 & -1 & 2\end{bmatrix} \sim I_3,$$
we have that $U = \mathbb{R}^3$ so that
$$\operatorname{proj}_U v = v.$$
NOTE: here v should be an element of $\mathbb{R}^3$.

- We have
$$\left(\begin{bmatrix}1\\-1\\2\\4\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-1\end{bmatrix} \right),$$
is a basis of U. Then, using the Gram-Schmidt process we have
$$\begin{aligned}
c_1 &= \begin{bmatrix}1\\-1\\2\\4\end{bmatrix},\\
c_2 &= \begin{bmatrix}1\\-1\\0\\0\end{bmatrix} - \tfrac{2}{22}\begin{bmatrix}1\\-1\\2\\4\end{bmatrix} = \begin{bmatrix}10/11\\-10/11\\-2/11\\-4/11\end{bmatrix},\\
c_3 &= \begin{bmatrix}0\\1\\0\\-1\end{bmatrix} - \tfrac{-5}{22}\begin{bmatrix}1\\-1\\2\\4\end{bmatrix} - \tfrac{-6/11}{20/11}\begin{bmatrix}10/11\\-10/11\\-2/11\\-4/11\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\2/5\\-1/5\end{bmatrix}
\end{aligned}$$
Then,
$$\operatorname{proj}_U v = \frac{c_1 \cdot v}{c_1 \cdot c_1} c_1 + \frac{c_2 \cdot v}{c_2 \cdot c_2} c_2 + \frac{c_3 \cdot v}{c_3 \cdot c_3} c_3.$$

Let's determine the orthogonal complements:

- We have
$$U^\perp = \left\{ \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} \right\}^\perp,$$
and $\underline{x} \in U^\perp$ if and only if
$$0 = \underline{x} \cdot \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} = x_1 + 3x_2 + 8x_3.$$
Hence,
$$U^\perp = \mathrm{span}_{\mathbb{R}} \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

- Since $U = \mathbb{R}^3$ then $U^\perp = \{0\}$.

- We have
$$U^\perp = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}^\perp,$$
so that $\underline{x} \in U^\perp$ if and only if
$$0 = \underline{x} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix} = x_1 - x_2 + 2x_3 + 4x_4,\ 0 = \underline{x} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = x_1 - x_2,\ 0 = \underline{x} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = x_2 - x_4.$$
Hence, we require that
$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \underline{x} = \underline{0}.$$
Since
$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix},$$
we have
$$U^\perp = \mathrm{span}_{\mathbb{R}} \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$