Math 110, Summer 2012: **Exam 2**

Instructor: George Melvin

Thursday, 8th August 2012 - 10.15am-12pm

Attempt at least THREE out of the following FOUR questions. You MAY ATTEMPT more than three questions: in this case, your best three answers will make up your overall score. Please CIRCLE BELOW THOSE QUESTIONS ATTEMPTED

1. This is a closed book exam. Please put away all your notes, textbooks, calculators and portable electronic devices and turn your mobile phones to 'silent' mode.
2. Explain your answers **CLEARLY** and **NEATLY**. State all theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and neatly.
3. Correct answers without appropriate justification will be treated with skepticism.
4. Write your name on this exam and any extra sheets you hand in.

Question 1:	/25
Question 2:	/25
Question 3:	/25
Question 4:	/25
Total:	/75

Name: SOLUTIONS

SID: ____________________________________

1. Let V be a finite dimensional $\mathbb{C}$-vector space and $L \in End_{\mathbb{C}}(V)$.

i) (2 pts) Define the representation ρ_L of $\mathbb{C}[t]$.

ii) (5 pts) Define what it means for $f \in \mathbb{C}[t]$ to be an annihilating polynomial of L. Define the minimal polynomial of L, μ_L. State the relationship between μ_L and *any* annihilating polynomial $f \in \mathbb{C}[t]$.

iii) (5 pts) Suppose that $f_1, f_2 \in \mathbb{C}[t]$ are annihilating polynomials of L, where

$$f_1 = (t-1)^4, \ f_2 = t^4 - 1.$$

Prove that $L = \mathrm{id}_V$.

iv) (5 pts) Suppose that $A \in Mat_n(\mathbb{C})$ is such that $\mu_A = (t-\alpha)^n$, for some $\alpha \in \mathbb{C}$. The Primary Decomposition Theorem tells us that

$$\mathbb{C}^n = \ker L^n, \text{ for some } L \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n).$$

Which L? Show that

$$A = \alpha I_n + N,$$

where $N \in Mat_n(\mathbb{C})$ is a nilpotent matrix of exponent $\eta(N) = n$.

v) (5 pts) Suppose that $A \in Mat_n(\mathbb{C})$ is such that $\mu_A = (t-\alpha)^n$, for some $\alpha \in \mathbb{C}$. Using 1iv), prove that the Jordan canonical form of A is

$$\begin{bmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & & \alpha \end{bmatrix}$$

vi) (3 pts) Consider the following matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in Mat_4(\mathbb{C}).$$

Determine the Jordan canonical form of A. Justify your answer.

Solution:

i) We have

$$\rho_L : \mathbb{C}[t] \to \mathrm{End}_{\mathbb{C}}(V) \ ; \ a_0 + a_1 t + ... + a_k t^k \mapsto a_0 \mathrm{id}_V + a_1 L + ... + a_k L^k.$$

ii) $f \in \mathbb{C}[t]$ is an annihilating polynomial if $f \in \ker \rho_L$ and f is nonzero. The minimal polynomial μ_L is the unique annihilating polynomial of L of minimal degree and with leading coefficient 1. For any annihilating polynomial f we have that μ_L divides f.

iii) We must have that μ_L divides both f_1 and f_2. Since

$$f_2 = (t-1)(t-\omega)(t+1)(t-\sqrt{-1})(t+\sqrt{-1}), \ f_1 = (t-1)^4,$$

we must have that $\mu_L = t - 1$. Hence, we have that $0_{\mathrm{End}_{\mathbb{C}}(V)} = \rho_L(\mu_L) = L - \mathrm{id}_V \implies L = \mathrm{id}_V$.

iv) As μ_A is an annihilating polynomial of A, then the Primary Decomposition Theorem states that

$$\mathbb{C}^n = \ker T_{(A-\alpha I_n)^n},$$

so that $L = T_{A-\alpha I_n}$. Hence, as $\ker L^n = \mathbb{C}^n$ we must have

$$0_n = [L^n]_{\mathcal{S}^{(n)}} = [L]^n_{\mathcal{S}^{(n)}} = [T_{A-\alpha I_n}]^n_{\mathcal{S}^{(n)}} = (A - \alpha I_n)^n.$$

Hence, the matrix $N = A - \alpha I_n$ is nilpotent. Moreover, it is not possible for $N^k = 0_n$, for any $k < n$, else then we would have $\mu_A = (t-\alpha)^k \neq (t-\alpha)^n$. Thus, the exponent of N is n. The claim follows.

v) Since N is nilpotent and has exponent n, then there must exist a vector $v \in \mathbb{C}^n$ such that $\text{ht}(v) = n$. Then, we have that

$$\mathcal{B} = (N^{n-1}v, N^{n-2}v, \dots, Nv, v),$$

is a basis of $\mathbb{C}^n$ and

$$[T_N]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

Hence, we must have

$$[T_{A-\alpha I_n}]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \implies [T_A - \text{id}_{\mathbb{C}^n}]_{\mathcal{B}} = [T_A]_{\mathcal{B}} - \alpha I_n = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

Thus, we have found a basis $\mathcal{B}$ such that

$$[T_A]_{\mathcal{B}} = \begin{bmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & & \alpha \end{bmatrix},$$

$$\implies \text{there exists } P \in \text{GL}_n(\mathbb{C}) \text{ such that } P^{-1}AP = \begin{bmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & & \alpha \end{bmatrix}.$$

vi) A satisfies

$$(A - I_4)^4 = 0_4, \text{ while } (A - I_4)^3 \neq 0_4,$$

so that $\mu_A = (t-1)^4$. Hence, by the previous problem we must have the Jordan canonical form of A is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. Let V be a finite dimensional $\mathbb{K}$-vector space, where $\mathbb{K}$ is a number field, and

$$B : V \times V \to \mathbb{K}.$$

i) (3 pts) Define what it means for B to be a symmetric $\mathbb{K}$-bilinear form.

ii) (2 pts) Suppose that B be a symmetric $\mathbb{K}$-bilinear form. Prove that $B(v, 0_V) = 0$, for any $v \in V$.

ii) (2 pts) Suppose that B is a symmetric $\mathbb{K}$-bilinear form, $E \subset V$ a nonempty subset. Define the B-complement $E^\perp$ of E in V.

iv) (5 pts) Suppose that B is a symmetric nondegenerate $\mathbb{K}$-bilinear form. Let $f \in \mathrm{End}_{\mathbb{K}}(V)$, $f^+ \in \mathrm{End}_{\mathbb{K}}(V)$ the adjoint of f (with respect to B). Prove that $\ker f^+ = (\mathrm{im} f)^\perp$.

v) (4 pts) Consider the $\mathbb{Q}$-bilinear form $B_A : \mathbb{Q}^2 \times \mathbb{Q}^2 \to \mathbb{Q}$, where

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}.$$

Show that B_A is symmetric and nondegenerate.

v) (5 pts) Determine the adjoint of f (with respect to B_A), f^+, where

$$f = T_C : \mathbb{Q}^2 \to \mathbb{Q}^2 \ ; \ \underline{x} \mapsto C\underline{x}, \quad C = \begin{bmatrix} 1 & -10 \\ 5 & -2 \end{bmatrix}.$$

(*Hint: it suffices to determine the matrix of f^+ with respect to some basis of $\mathbb{Q}^2$*)

vi) (4 pts) **Without** calculating $\ker f$ or using the definition of 'injective', show that f is injective. (*Hint: use f^+*)

Solution:

i) B is symmetric if $B(u, v) = B(v, u)$, for every $u, v \in V$. B is a $\mathbb{K}$-bilinear form if

- for every $u, v, w \in V, \lambda \in \mathbb{K}$, $B(u + \lambda v, w) = B(u, w) + \lambda B(v, w)$,
- for every $u, v, w \in V, \lambda \in \mathbb{K}$, $B(u, v + \lambda w) = B(u, v) + \lambda B(v, w)$.

ii) Let $v \in V$. We have

$$B(v, 0_V) = B(v, 0_V + 0_V) = B(v, 0_V) + B(v, 0_V) \implies B(v, 0_V) = 0.$$

iii)

$$E^\perp = \{v \in V \mid B(v, e) = 0, \text{ for every } e \in E\}.$$

iv) Recall that, for every $u, v \in V$,

$$B(f(u), v) = B(u, f^+(v)).$$

Let $v \in \ker f^+$. Then, for any $u \in V$, we have

$$0 = B(u, f^+(v)) = B(f(u), v) \implies v \in (\mathrm{im} f)^\perp.$$

Conversely, let $v \in (\mathrm{im} f)^\perp$. Then, for any $u \in V$,

$$0 = B(f(u), v) = B(u, f^+(v)).$$

Hence, as B is nondegenerate we must have $f^+(v) = 0_V$, so that $v \in \ker f^+$. Thus, $\ker f^+ = (\mathrm{im} f)^\perp$.

v) As A is a symmetric and invertible matrix then we must have that B_A is symmetric and nondegenerate, using results from class.

vi) We know that

$$[f^+]_{\mathcal{S}^{(2)}} = [B_A]_{\mathcal{S}^{(2)}}^{-1}[f]_{\mathcal{S}^{(2)}}^t[B_A]_{\mathcal{S}^{(2)}} = A^{-1}C^tA = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 39 & -67 \\ 24 & -40 \end{bmatrix} \stackrel{def}{=} C'.$$

Thus, $f^+ = T_{C'}$.

vii) Since C' is invertible ($\det C' = 48$), then f^+ is injective, so that $(\mathrm{im} f)^\perp = \{0\}$ by iv). Hence, $\mathbb{Q}^2 = \mathrm{im} f \oplus (\mathrm{im} f)^\perp = \mathrm{im} f$ and f is surjective. Hence, since f is an endomorphism it must also be injective.

3. Throughout this problem we will assume that B is a symmetric nondegenerate $\mathbb{R}$-bilinear form on the finite dimensional $\mathbb{R}$-vector space V.

i) (3 pts) State the polarisation identity.

ii) (5 pts) Using the polarisation identity, prove that there exists nonzero $v \in V$ such that $B(v, v) \neq 0$. Deduce that there exists $w \in V$ such that $B(w, w) = 4$.

iii) (6 pts) Consider the $\mathbb{R}$-bilinear form

$$B : Mat_2(\mathbb{R}) \times Mat_2(\mathbb{R}) \to \mathbb{R} \; ; \; (A, B) \mapsto \mathrm{tr}(A^t X B), \quad \text{where } X = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Determine the matrix of B with respect to the basis $\mathcal{S} = (e_{11}, e_{12}, e_{21}, e_{22})$, $[B]_{\mathcal{S}} \in Mat_4(\mathbb{R})$. Deduce that B is symmetric and nondegenerate. Justify your answer.

iv) (8 pts) Determine P such that

$$P^t[B]_{\mathcal{S}}P = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}, \; d_1, d_2, d_3, d_4 \in \{1, -1\}.$$

v) (3 pts) Determine $A \in Mat_2(\mathbb{R})$ such that $B(A, A) = 4$.

Solution:

i) For every $u, v \in V$, we have

$$B(u, v) = \frac{1}{2}(B(u + v, u + v) - B(u, u) - B(v, v)).$$

ii) Suppose that $B(v, v) = 0$, for every $v \in V$. Then, since B is nondegenerate, B is nonzero. Hence, we know that there exists $u_0, v_0 \in V$ such that $B(u_0, v_0) \neq 0$. However, by our assumption and the polarisation identity, we would then have

$$0 \neq B(u_0, v_0) = \frac{1}{2}(B(u_0 + v_0, u_0 + v_0) - B(u_0, u_0) - B(v_0, v_0)) = 0 - 0 - 0 = 0,$$

which is a contradiction. Hence, there must exist some $v \in V$ such that $B(v, v) \neq 0$. Suppose that $B(v, v) > 0$. If we let $w = \frac{2v}{\sqrt{B(v,v)}}$, then

$$B(w, w) = B(\frac{2v}{\sqrt{B(v, v)}}, \frac{2v}{\sqrt{B(v, v)}}) = 4.$$

If $B(v, v) < 0$, let $w = \frac{2v}{\sqrt{-B(v,v)}}$.

iii) We have that

$$A = [B]_{\mathcal{S}} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Since this matrix is symmetric and invertible then B is symmetric and nondegenerate.

iv) We have

$$\underline{x}^t A \underline{x} = -2x_1x_3 - 2x_2x_4 = \frac{1}{2}(x_1 - x_3)^2 - \frac{1}{2}(x_1 + x_3)^2 + \frac{1}{2}(x_2 - x_4)^2 - \frac{1}{2}(x_2 + x_4)^2.$$

Let

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

then

$$P^t AP = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}.$$

v) Let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

then $B(A, A) = 4$.

4. Let V be a finite dimensional $\mathbb{R}$-vector space.

i) (2 pts) Let B be a symmetric bilinear form on V. Define what it means for B to be an inner product.

ii) (4 pts) Suppose that B is an inner product on V. Prove that B is nondegenerate.

iii) (2 pts) Let $(V_1, \langle , \rangle_1), (V_2, \langle , \rangle_2)$ be Euclidean spaces, $f \in \mathrm{Hom}_{\mathbb{R}}(V_1, V_2)$. Define what it means for f to be a Euclidean morphism.

iv) (4 pts) Let $(V_1, \langle , \rangle_1), (V_2, \langle , \rangle_2)$ be Euclidean spaces, $f : V_1 \to V_2$ a Euclidean morphism. Prove: if $\dim V_1 = \dim V_2$ then f is an isomorphism.

Consider the symmetric bilinear form

$$B_A : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \ ; \ (\underline{x}, \underline{y}) \mapsto \underline{x}^t A \underline{y}, \quad A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

iv) (4 pts) Show that B_A is an inner product on $\mathbb{R}^3$.

v) (5 pts) Determine a Euclidean isomorphism

$$f : (\mathbb{R}^3, B_A) \to \mathbb{E}^3.$$

vi) (4 pts) **Without** using the Gram-Schmidt process, determine an orthonormal basis of $(\mathbb{R}^3, B_A)$. (*Hinit: use 4v)*)

Solution:

i) B is an inner product if, for every $v \in V$ we have $B(v, v) \geq 0$ and $B(v, v) = 0$ if and only if $v = 0_V$.

ii) Suppose that $v \in V$ is such that $B(u, v) = 0$, for every $u \in V$. Then, in particular, we have $B(v, v) = 0$, so that $v = 0_V$, since B is an inner product. Hence, B is nondegenerate. item[iii)] We must have, for every $u, v \in V_1$,

$$\langle f(u), f(v) \rangle_2 = \langle u, v \rangle_1.$$

iv) Suppose that f is a Euclidean morphism and $\dim V_1 = \dim V_2$. Then, f is injective: indeed, if $f(v) = 0_{V_2}$ then

$$0 = \langle f(v), f(v) \rangle_2 = \langle v, v \rangle_1 \implies v = 0_{V_1}.$$

Hence, f is injective and therefore bijective, as $\dim V_1 = \dim V_2$.

v) We have

$$\underline{x}^t A \underline{x} = x_1^2 - 2x_1x_2 + 3x_2^2 + 4x_2x_3 + 5x_3^2 = (x_1 - x_2)^2 + 2(x_2 + x_3)^2 + 3x_3^2.$$

Hence, the signature of B_A is 3, so that B_A must be an inner product.

vi) Consider the matrix

$$Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{3} \end{bmatrix},$$

then $T_Q : \mathbb{R}^3 \to \mathbb{R}^3$ is an isomorphism (as Q is invertible) and, for any $\underline{x}, \underline{y} \in \mathbb{R}^3$,

$$(Q\underline{x}) \cdot (Q\underline{y}) = \underline{x}^t Q^t Q \underline{y} = \underline{x}^t A \underline{y} = B_A(\underline{x}, \underline{y}),$$

since $Q^t Q = A$.

vii) Let

$$P = Q^{-1} = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Then, $T_P = T_Q^{-1}$ and, since T_Q is a Euclidean isomorphism, then $B_A(Pe_i, Pe_j) = 0$, if $i \neq j$, and $B_A(Pe_i, Pe_i) = 1$. Hence, the set

$$\{Pe_1, Pe_2, Pe_3\},$$

determines an orthonormal basis of $(\mathbb{R}^3, B_A)$.