Math 110, Summer 2012: **Exam 1**

Instructor: George Melvin

Monday, 16th July 2012 - 10.15am-12pm

Attempt at least THREE out of the following FIVE questions. You MAY ATTEMPT more than three questions: in this case, your best three answers will make up your overall score. Please CIRCLE BELOW THOSE QUESTIONS ATTEMPTED

1. This is a closed book exam. Please put away all your notes, textbooks, calculators and portable electronic devices and turn your mobile phones to 'silent' mode.
2. Explain your answers **CLEARLY** and **NEATLY**. State all theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and neatly.
3. Correct answers without appropriate justification will be treated with skepticism.
4. Write your name on this exam and any extra sheets you hand in.

Question 1:	/25
Question 2:	/25
Question 3:	/25
Question 4:	/25
Question 5:	/25
Total:	/75

Name: ________________________________

SID: ________________________________

1. Let V be a $\mathbb{K}$-vector space, for some number field $\mathbb{K}$. Let $E \subset V$ be a nonempty subset of V.

i) (4 pts) Define what it means for E to be linearly independent (over $\mathbb{K}$). Define what it means for E to be linearly dependent (over $\mathbb{K}$).

ii) (3 pts) Suppose that E is linearly independent and let $F \subset E$ be a nonempty subset. Prove that F is linearly independent.

iii) (5 pts) Suppose that E is linearly dependent. Prove that there exists $v \in E$ such that v can be written as a linear combination

$$v = c_1 v_1 + \dots + c_k v_k, \quad \text{with } c_i \in \mathbb{K}, v_i \in E.$$

iv) (7 pts) Suppose that $E = \{e_1, \dots, e_n\}$ is linearly independent. Let $w \in V$ be such that $w \notin \mathrm{span}_{\mathbb{K}} E$. Prove that $E \cup \{w\}$ is linearly independent.

v) (6 pts) Show that

$$E = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\} \subset Mat_2(\mathbb{Q}),$$

is linearly independent and extend E to a basis of $Mat_2(\mathbb{Q})$.

2. i) (3 pts) Let $\mathcal{B} = (b_1, \dots, b_n) \subset V$ be an ordered subset of the $\mathbb{K}$-vector space V. Define what it means for $\mathcal{B}$ to be an ordered basis of V (*You can use ANY definition here.*)

ii) (2 pts) Suppose that $E \subset V$ is a linearly independent subset of a finite dimensional $\mathbb{K}$-vector space V. What is the allowed possible size of E?

iii) (6 pts) Suppose that V is a $\mathbb{K}$-vector space such that $\dim_{\mathbb{K}} V = n$. Let $E \subset V$ be a linearly independent subset of size $|E| = n$. Prove that $\operatorname{span}_{\mathbb{K}} E = V$. (*Hint: Use a 'proof by contradiction' argument.*)

iv) (6 pts) Consider the ordered subset

$$\mathcal{B} = (f_1, f_2, f_3) \subset \mathbb{Q}^{\{1,2,3\}} = \{f : \{1,2,3\} \to \mathbb{Q}\},$$

where

$$f_1(1) = 1, f_1(2) = 0, f_1(3) = -1,\ f_2(1) = 1, f_2(2) = 0, f_2(3) = 1,\ f_3(1) = 0, f_3(2) = 1, f_3(3) = 1.$$

Prove that $\mathcal{B}$ is linearly independent. Deduce that $\mathcal{B}$ is a basis of $\mathbb{Q}^{\{1,2,3\}}$.

v) (5 pts) Let $\mathcal{S} = (e_1, e_2, e_3) \subset \mathbb{Q}^{\{1,2,3\}}$ be the standard ordered basis of $\mathbb{Q}^{\{1,2,3\}}$. Determine the change of coordinate matrix $P_{\mathcal{B} \leftarrow \mathcal{S}}$.

vi) (3 pts) Suppose that $f \in \mathbb{Q}^{\{1,2,3\}}$ is such that

$$[f]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

Is $f \in \operatorname{span}_{\mathbb{Q}}\{f_1, f_3\}$? Justify your answer.

3. i) (6 pts) Define the image im$f$ of a linear morphism $f : V \to W$ and the rank of f, rank$f$. Define the rank of an $m \times n$ matrix $A \in Mat_{m,n}(\mathbb{K})$, rank$A$.

ii) (7 pts) Prove: if rank$f = \dim V$ then f is surjective.

iii) (5 pts) Prove: if $A \in Mat_{m,n}(\mathbb{K}), B \in Mat_{n,p}(\mathbb{K})$ and rank$A = r$, rank$B = s$, then rank$AB \leq r$.

iv) (7 pts) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Determine $r = \text{rank}A$ and find $P, Q \in \mathrm{GL}_3(\mathbb{Q})$ such that

$$Q^{-1}AP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

4. i) (4 pts) Let $f \in \mathrm{End}_{\mathbb{C}}(V)$, with V a finite dimensional $\mathbb{C}$-vector space. Define what it means for $\lambda \in \mathbb{C}$ to be an eigenvalue of f. Define the geometric and algebraic multiplicity of λ.

ii) (4 pts) Let $f \in \mathrm{End}_{\mathbb{C}}(V)$, with V a finite dimensional $\mathbb{C}$-vector space. Define what it means for f to be diagonalisable. Give a criterion for f to be diagonalisable using the notions of geometric and algebraic multiplicity of eigenvalues.

iii) (7 pts) Let $f \in \mathrm{End}_{\mathbb{C}}(V)$, where $\dim V = 7$. Suppose that f is non-surjective, diagonalisable and such that $\dim \mathrm{im} f = 1$. Prove that f admits exactly one nonzero eigenvalue λ and that $E_\lambda = \mathrm{im} f$, where E_λ is the λ-eigenspace.

Consider the endomorphism

$$f : Mat_2(\mathbb{C}) \to Mat_2(\mathbb{C}) \; ; \; A \mapsto A + A^t,$$

where A^t is the transpose of A.

iv) (4 pts) Determine the eigenvalues of f and their algebraic multiplicities.

v) (6 pts) Prove that f is diagonalisable and find a basis $\mathcal{B} \subset Mat_2(\mathbb{C})$ such that $[f]_{\mathcal{B}}$ is diagonal.

For iv)-v) you may want to use the standard ordered basis

$$\mathcal{S} = (e_{11}, e_{12}, e_{21}, e_{22}) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \subset Mat_2(\mathbb{C}).$$

5. Consider the following endomorphism

$$L_A : Mat_n(\mathbb{C}) \to Mat_n(\mathbb{C}) \ ; \ B \mapsto AB, \quad \text{where } A \in Mat_n(\mathbb{C}).$$

i) (4 pts) Define what it means for L_A to be nilpotent. Define what it means for A to be nilpotent.

ii) (2 pts) Define the exponent of L_A, $\eta(L_A)$. Define the exponent of A, $\eta(A)$.

iii) (4 pts) Prove: A is nilpotent if and only if L_A is nilpotent.

Now suppose that $n = 2$ **and** $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$**.**

iv) (2 pts) Using iii) deduce that L_A is nilpotent. What is $\eta(L_A)$?

v) (3 pts) Let $\mathcal{S} = (e_{11}, e_{12}, e_{21}, e_{22})$ be the standard ordered basis of $Mat_2(\mathbb{C})$. Determine $X = [L_A]_{\mathcal{S}}$.

vi) (7 pts) Determine an ordered basis $\mathcal{B} = (b_1, b_2, b_3, b_4) \subset Mat_2(\mathbb{C})$ such that $[L_A]_{\mathcal{B}}$ is a block diagonal matrix, each block being a 0-Jordan block.

vii) (3 pts) Determine the partition associated to X, $\pi(X)$. Is X similar to the following matrix

$$Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}?$$

Justify your answer.