Worksheet 7/25. Math 110, Summer 2012

*An asterisk * denotes a harder problem. Speak to your neighbours, these problems should be discussed.*

Nondegenerate bilinear forms

Consider the following bilinear forms

i) $B_1 : \mathbb{Q}^3 \times \mathbb{Q}^3 \to \mathbb{Q}$; $(u, v) \mapsto u_1v_3 + 5u_2v_2 - 6u_3v_1$.

iii) $B_2 : Mat_{3,2}(\mathbb{Q}) \times Mat_{3,2} \to \mathbb{Q}$; $(X, Y) \mapsto \text{tr}(XY^t)$.

1. Prove that if B is nondegenerate and $v \in V$, then there exists $v' \in V$ such that $B(v, v') \neq 0$. Deduce that there is some $w \in V$ such that $B(v, w) = 1$.

2. Consider the bilinear form

$$B : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C} \; ; \; (\underline{x}, \underline{y}) \mapsto x_1y_1 + x_1y_2.$$

Show that B is degenerate by finding an explicit $\underline{x}_0 \in \mathbb{C}^2$ such that $B(\underline{y}, \underline{x}_0) = 0$, for every $\underline{y} \in \mathbb{C}^2$.

3. Which of B_1, B_2 above are nondegenerate? If B_i is nondegenerate determine $v_i \in V$ (here $V = \mathbb{Q}^3$ or $V = Mat_{3,2}(\mathbb{Q})$) such that

$$B(v_i, e_1 + e_2) = 1,$$

where we are assuming $e_1, e_2 \in \mathcal{S}$, with $\mathcal{S}$ the usual standard ordered basis of V (eg, if $V = Mat_{3,2}(\mathbb{Q})$ then $e_1 = e_{11}, e_2 = e_{12}$ etc).

Adjoints

4. Determine the adjoint of f for the following morphisms f (with respect to the appropriate bilinear form B_i above).

i) $f : \mathbb{Q}^3 \to \mathbb{Q}^3$; $\underline{x} \mapsto A\underline{x}$, where

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & 4 & -3 \end{bmatrix}.$$

ii) $f : Mat_{3,2}(\mathbb{Q}) \to Mat_{3,2}(\mathbb{Q})$; $X \mapsto BX$, where

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What is the adjoint of f in i) with respect to 'dot product'?