Worksheet 7/24. Math 110, Summer 2012

*An asterisk * denotes a harder problem. Speak to your neighbours, these problems should be discussed.*

Bilinear forms

1. Which of the following functions are bilinear forms

   i) $B : \mathbb{Q}^3 \times \mathbb{Q}^3 \to \mathbb{Q}\ ;\ (u, v) \mapsto u_1 v_3 + 5u_2 v_2 - 6u_3 v_1$.

   ii) $B : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}\ ;\ (u, v) \mapsto u_1^2 - 2u_1 v_3 + v_2 - u_3 v_1$.

   iii) $B : Mat_{3,2}(\mathbb{Q}) \times Mat_{3,2} \to \mathbb{Q}\ ;\ (X, Y) \mapsto \mathrm{tr}(XY^t)$.

For those functions that are bilinear forms determine the matrix $[B]_{\mathcal{S}}$, where $\mathcal{S}$ is the appropriate standard basis. Which of the bilinear forms are symmetric/antisymmetric/neither? Which are nondegenerate?

2. Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$, $A = [B]_{\mathcal{B}}$. Prove:

   - B is symmetric if and only if A is symmetric (so $A = A^t$).
   - B is antisymmetric if and only if A is antisymmetric (so $A = -A^t$).

3. Let $\alpha, \beta \in V^*$. Consider the function

$$B_{\alpha,\beta} : V \times V \to \mathbb{K}\ ;\ (u, v) \mapsto \alpha(u)\beta(v).$$

* Is B symmetric/antisymmetric/neither?

4. Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$, $\mathcal{B} \subset V$ be an ordered basis. Is it necessarily true that if $[B]_{\mathcal{B}}$ is a diagonal matrix then $[B]_{\mathcal{C}}$, for any basis $\mathcal{C} \subset V$, is diagonalisable?

5. Let $A \in Mat_2(\mathbb{R})$ be symmetric. Is A diagonalisable? If $A \in Mat_2(\mathbb{Q})$ is symmetric then is A diagonalisable?

6. Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ be nondegenerate. Prove that, if $B(u, v) = 0$, for every $v \in V$, then $u = 0_V$.