Worksheet 7/18. Math 110, Summer 2012

*An asterisk * denotes a harder problem. Speak to your neighbours, these problems should be discussed.*

Minimal polynomial

1. Suppose that $A \in Mat_n(\mathbb{C})$ such that

$$\mu_A = (t-1)^2.$$

Which of the following polynomial relations can A satisfy?

- $A^3 - 2A^2 + A = 0_n$,
- $A^3 - 3A^2 + 3A - I_n = 0_n$,
- $A^2 - 3A + 2 = 0_n$,
- $A^5 - 3A^4 + 5A - 2 = 0_n$.

2. Prove that the minimial polynomial of an endomorphism L is unique: ie, show that if $f \in \ker \rho_L$ and $\deg f = \deg \mu_L$ and the leading coefficient of f is 1, then $f = \mu_L$.

3. Determine the minimal polynomials of the following matrices (you will want to use that $\chi_A \in \ker \rho_A$):

- $A = \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix}$,
- $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,
- $A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

Are these matrices diagonalisable?

4. True or false: if $L \in \text{End}_{\mathbb{C}}(\mathbb{C}^n)$ is an endomorphism and $f \in \ker \rho_L$ is such that $\deg f = n$ and f has leading entry 1, then $\chi_L = f$. Prove your assertion if you think the statement is TRUE; provide a counterexample if you think the statement is FALSE.

5. Let $L \in \text{End}_{\mathbb{C}}(V)$ and $n = \dim V$. Prove that χ_L divides μ_L^n.

6. Suppose that $A \in Mat_n(\mathbb{C})$ is invertible, diagonalisable and such that

$$A^6 - 4A^5 + 6A^4 - 4A^3 + A^2 = 0_n.$$

Prove that $A = I_n$.