Worksheet 7/3. Math 110, Summer 2012

*An asterisk * denotes a harder problem. Speak to your neighbours, these problems should be discussed.*

Diagonalisation

1. Which of the following matrices are diagonalisable? Justify your answer carefully.

1. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

For those matrices A that are diagonalisable find an invertible matrix P such that $P^{-1}AP$ is diagonal. Which of the above matrices are invertible?

2. Let $A \in Mat_6(\mathbb{C})$ be such that $\text{rank}(A) = 2$, and $Ae_1 = -e_1$, where e_1 is the first standard basis vector of $\mathbb{C}^6$. Is it necessarily true that A is diagonalisable? What if you know that A is similar to the matrix

$$B = \begin{bmatrix} -I_2 & 0 \\ 0 & C \end{bmatrix},$$

where $C \in Mat_4(\mathbb{C})$, is A necessarily diagonalisable? Justify your answer. What are the eigenvalues of A in this case?

3. Let $A \in Mat_{10}(\mathbb{C})$ and suppose that A is invertible. Suppose that $A^{11} = A$. What are the possible eigenvalues of A? Is A diagonalisable? Justify your answer.

Invariant subspaces

4. Prove: let $\mathcal{B} \subset V$ be an ordered basis such that the matrix $[f]_{\mathcal{B}}$ is block diagonal, with two blocks. Then, $V = U \oplus W$, where both U and W are f-invariant.

5. Consider the matrix

$$A = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

where $t \in \mathbb{R}$. Prove that if $U \subset \mathbb{R}^2$ is A-invariant then $U = \{\underline{0}\}$. In particular, there are no nonzero B-invariant subspaces of $\mathbb{R}^2$, where

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Prove that there exists B-invariant subspaces of $\mathbb{C}^2$, however.