Math 110, Summer 2012 Short Homework 8 (SOME) SOLUTIONS

Due Monday 7/23, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

0. Was this homework assignment too easy/too difficult/about right? Any other comments are welcome.

Calculations

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

i) Determine $\chi_A(t)$.

ii) Show that A is NOT diagonalisable without appealing to algebraic/geometric multiplicities. (*Use the diagonalisablity criterion involving the minimal polynomial. What must the minimal polynomial be if A were diagonalisable?*

iii) Determine the subspaces

$$U_1 = \ker T_{(A-2I_4)^3},\ U_2 = \ker T_A,$$

and a basis $\mathcal{C} = (c_1, c_2, c_3) \subset U_1$.

iv) Consider the endomorphism

$$f : U_1 \to U_1 \ ;\ u \mapsto Au - 2u.$$

Determine $B = [f]_{\mathcal{C}}$.

v) Show that B is nilpotent and find a basis $\mathcal{C}' \subset U_1$ such that $[f]_{\mathcal{C}'}$ is block diagonal, each block being a 0-Jordan block.

vi) Determine an invertible matrix $P \in GL_4(\mathbb{C})$ such that

$$P^{-1}AP = J,$$

where J is the Jordan form of A.

Solution:

i) We have

$$\chi_A(t) = t(t-2)^3.$$

ii) Since

$$A(A-2I_4) = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0_4,$$

then the minimal polynomial $\mu_A \neq t(t-2)$, so that A can't be diagonalisable (A is diagonalisable if and only if μ_A is a product of distinct linear factors).

iii) We have

$$(A-2I_4)^3 = \begin{bmatrix} -4 & 4 & 0 & 0 \\ 4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so that

$$U_1 = \ker T_{(A-2I_4)^3} = \text{span}_{\mathbb{C}}\{e_1 + e_2, e_3, e_4\}.$$

Also,

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and we have

$$U_2 = \ker T_A = \text{span}_{\mathbb{C}}\{e_1 - e_2\}.$$

In particular, we see that we can take

$$\mathcal{C} = (e_1 + e_2, e_3, e_4) = (c_1, c_2, c_3) \subset \mathbb{C}^4.$$

iv) We need to determine

$$B = [[f(c_1)]_{\mathcal{C}}[f(c_2)]_{\mathcal{C}}[f(c_3)]_{\mathcal{C}}] = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

v) Since $B^2 = 0_3$ we see that B is nilpotent. Now, using the method of section 2.3 to find a matrix P such that $P^{-1}BP$ is block diagonal and each block is a 0-Jordan block, we can take

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

That is we have a basis $\mathcal{B}' = (e_1, e_2, e_2 - e_3) \subset \mathbb{C}^3$ such that

$$[T_B]_{\mathcal{B}'} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Sicne U_1 is isomorphic to $\mathbb{C}^3$ via the $\mathcal{C}$-coordinate morphsim, we see that with respect to the basis

$$\mathcal{C}' = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right) \subset U_1,$$

we have

$$[f]_{\mathcal{C}'} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

vi) We take the union of $\mathcal{C}'$ and a basis $\mathcal{C}'' \subset U_2$. Since U_2 is one dimensional (as $\mathbb{C}^4 = U_1 \oplus U_2$) then we can take a basis $\mathcal{C}'' = (e_1 - e_2)$. Hence, if we set

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

then

$$P^{-1}AP = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proofs

2. Let $f \in \mathrm{End}_{\mathbb{C}}(V)$ be such that $\chi_f(t) = t^{\dim V}$. Prove that f is nilpotent.

Solution: We use the Cayley-Hamilton Theorem: $\chi_f \in \ker \rho_f$, where ρ_f is the representation of $\mathbb{C}[t]$ defined by f. This tells us that

$$f^{\dim V} = 0_{\mathrm{End}_{\mathbb{C}}(V)},$$

so that f is nilpotent.

3. Let $A \in Mat_7(\mathbb{C})$ be an invertible matrix and such that

$$A^7 - 6A^4 - 6A^6 + 11A^5 = 0_7 \in Mat_7(\mathbb{C}).$$

Prove that A is diagonalisable. (*Hint: It may be useful to know that* 1 *is an eigenvalue of* A. *You will need to perform the division algorithm (ie, long division of polynomials) at some point in your solution.*)

Solution: Since A is invertible then A^{-1} exists and we can multiply the above polynomial on both sides by A^{-4}. Then, we obtain

$$A^3 - 6I_7 - 6A^2 + 11A = 0_7.$$

Hence,

$$f = t^3 - 6t^2 + 11t - 6 \in \ker \rho_A,$$

where ρ_A is the representation of $\mathbb{C}[t]$ defined by A. You can check that

$$f = (t-1)(t-2)(t-3),$$

using long division and the hint that $t = 1$ is a root of $t^4 f$. Then, we must have that μ_A divides f. Since f is product of distinct linear factors, the same must be true of μ_A: if μ_A has repeated linear factors then these would also appear in f. Hence, A is diagonalisable, by a result from section 2.5.