Math 110, Summer 2012 Short Homework 7

Due Monday 7/12, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Consider the $\mathbb{C}$-vector space $\mathbb{C}_3[t]$ consisting of polynomials with $\mathbb{C}$-coefficients that have degree at most 3. We have $\dim_{\mathbb{C}} \mathbb{C}_3[t] = 4$. Consider the $\mathbb{C}$-linear endomorphism

$$D : \mathbb{C}_3[t] \to \mathbb{C}_3[t] \; ; \; f \mapsto \frac{df}{dt}.$$

a) Show that D is a nilpotent endomorphism and determine the exponent of D, $\eta(D)$.

b) For each k, $0 \leq k \leq \eta(D)$, determine

$$H_k = \{f \in \mathbb{C}_3[t] \mid \mathrm{ht}(f) \leq k\},$$

and determine $\dim H_k = m_k$.

c) Recall the algorithm from Section 2.3 used to determine a basis of V, given a nilpotent endomorphism $g \in \mathrm{End}_{\mathbb{C}}(V)$. Using this algorithm find an ordered basis $\mathcal{B}$ of $\mathbb{C}_3[t]$ such that $[D]_{\mathcal{B}}$ is block diagonal matrix, each block being a 0-Jordan block.

(*Hint: there is only one* 0*-Jordan block.*)

Solution:

a) You can easily check that D^4 is the zero morphism: for any $f \in \mathbb{C}_3[t]$ we have $D^4(f) = 0_{\mathbb{C}_3[t]}$. Hence, $\eta(D) = 4$.

b) For each $0 \leq k \leq 4$ we have

$$H_k = \{f \in \mathbb{C}_3[t] \mid D^k(f) = 0\} = \{f \in \mathbb{C}_3[t] \mid \deg f \leq k - 1\}.$$

Then, $\dim H_k = \dim \mathbb{C}_{k-1}[t] = k$.

c) We proceed through the algorithm described in the notes:[1]

We have

$$\mathbb{C}_3[t] = H_4 = H_3 \oplus G_4 = \mathrm{span}\{1, t, t^2\} \oplus \mathrm{span}\{t^3\},$$

where we have used that $t^3 \notin H_3$ (so that $G_4 = \mathrm{span}\{t^3\}$) and $H_3 = \mathrm{span}\{1, t, t^2\}$. Thus, we have $z_1 = t^3$. Then, since $\mathrm{ht}(t^3) = 4$ we have that $\{t^3, D(t^3), D^2(t^3), D^3(t^3)\}$ is linearly independent therefore must be a basis of $\mathbb{C}_3[t]$. If we let $\mathcal{B} = (6, 6t, 3t^2, t^3)$ then

$$[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in Mat_2(\mathbb{C}),$$

and consider the endomorphism

$$R_A : Mat_2(\mathbb{C}) \to Mat_2(\mathbb{C}) \; ; \; B \mapsto BA.$$

[1] However, this is not required since the exponent of D is 4. This means the largest number appearing in $\pi(D)$ is 4. The only partition of 4 for which 4 appears is the partition $\pi(D) : 4$. Thus, we necessarily must have that $(D^3 f, D^2 f, Df, f)$ is a Jordan basis, where f is any vector of height 4.

a) Show that R_A is a nilpotent endomorphism and determine the exponent of R_A, $\eta(R_A)$.

b) For each k, $0 \leq k \leq \eta(R_A)$, determine

$$H_k = \{B \in Mat_2(\mathbb{C}) \mid \mathrm{ht}(B) \leq k\},$$

and determine $\dim H_k = m_k$.

c) As in 1c) above, determine an ordered basis $\mathcal{B} \subset Mat_2(\mathbb{C})$ such that $[R_A]_{\mathcal{B}}$ is a block diagonal matrix, each block being a 0-Jordan block.

(*Hint: there is more than one* 0*-Jordan block in this case.*)

Solution:

a) You can check that $R_A^2 = 0 \in \mathrm{End}_{\mathbb{C}}(Mat_2(\mathbb{C}))$ so that $\eta(R_A) = 2$.

b) We have, for each $0 \leq k \leq 2$ that

$$H_k = \ker R_A^k = \{B \in Mat_2(\mathbb{C}) \mid R_A^k(B) = 0\}.$$

Thus, we have

$$H_0 = \{0\},\ H_1 = \{B \in Mat_2(\mathbb{C}) \mid BA = 0\} = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{C} \right\}, H_2 = Mat_2(\mathbb{C}).$$

Hence,

$$m_0 = 0,\ m_1 = 2,\ m_2 = 4.$$

c) We have

$$Mat_2(\mathbb{C}) = H_2 = H_1 \oplus G_2 = H_1 \oplus \mathrm{span}\{e_{11}, e_{21}\},$$

so take $z_1 = e_{11}, z_2 = e_{21}$. Then, we know that $\{R_A(e_{11}), e_{11}, R_A(e_{21}), e_{21}\}$ is linearly independent and hence must be a basis. Thus, if we define

$$\mathcal{B} = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right),$$

we must have

$$[R_A]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proofs

3. Let $f \in \mathrm{End}_{\mathbb{C}}(V)$, where V is a finite dimensional $\mathbb{C}$-vector space. Denote the eigenvalues of f by $\lambda_1, \dots, \lambda_k$. Prove: f is diagonalisable if and only if, for every i, the algebraic multiplicity of λ_i is equal to the geometric multiplicity of λ_i.

(*Looking at Proposition 2.1.14 and its proof may help here.*)

4. Let $f \in \mathrm{End}_{\mathbb{C}}(V)$, where $\dim V = n$, and suppose that there is an ordered basis $\mathcal{B} = (b_1, \dots, b_n)$ of V such that

$$[f]_{\mathcal{B}} = \begin{bmatrix} A & B \\ 0_{n-k,k} & C \end{bmatrix}.$$

Prove that $U = \mathrm{span}_{\mathbb{C}}\{b_1, \dots, b_k\}$ is f-invariant.

Solution: Let $u \in \mathrm{span}_{\mathbb{C}}\{b_1, \dots, b_k\}$. Then,

$$u = c_1 b_1 + \dots + c_k b_k,\ c_1, \dots, c_k \in \mathbb{C}.$$

Thus, we have

$$[f]_{\mathcal{B}}[u]_{\mathcal{B}} = \begin{bmatrix} A & B \\ 0_{n-k,k} & C \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} A\underline{c} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $\underline{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{C}^k$. Then, $A\underline{c} \in \mathbb{C}^k$ since $A \in Mat_k(\mathbb{C})$. Hence,

$$[f(u)]_{\mathcal{B}} = [f]_{\mathcal{B}}[u]_{\mathcal{B}} \in \operatorname{span}_{\mathbb{C}}\{[b_1]_{\mathcal{B}}, \dots, [b_k]_{\mathcal{B}}\},$$

so that $f(u) \in \operatorname{span}_{\mathbb{C}}\{b_1, \dots, b_k\}$, since the $\mathcal{B}$-coordinate morphism is an isomorphism.

(*Short Homework 6*) 2. Consider the matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in Mat_3(\mathbb{C}).$$

Show that the subspace

$$U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0 \right\} \subset \mathbb{C}^3,$$

is B-invariant and that 1 is an eigenvalue of B. Show that $E_1 \cap U = \{0_{\mathbb{C}^3}\}$. Find a B-invariant subspace $W \subset V$ such that

$$V = W \oplus U.$$

Justify your answer.

Solution: Let $\underline{x} \in U$ so that $x_1 + x_2 + x_3 = 0$. Then,

$$B\underline{x} = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix} \left(= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right),$$

and $y_1 + y_2 + y_3 = x_3 + x_1 + x_2 = 0$, so that $B\underline{x} \in U$. Hence, U is B-invariant.

You can check that

$$B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

so that 1 is an eigenvalue of B.

Let $z \in E_1 \cap U$. Then, $Bz = z$ and $z_1 + z_2 + z_3 = 0$. As $Bz = z$ we must have

$$\begin{bmatrix} z_3 \\ z_1 \\ z_2 \end{bmatrix} = Bz = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix},$$

so that $z_3 = z_1 = z_2$. Now, as $z_1 + z_2 + z_3 = 0$ this can only occur if $z_1 = z_2 = z_3 = 0$. Hence, $z = 0$.

As U is 2 dimensional (you can easily check this) and $E_1 + U = E_1 \oplus U \subset \mathbb{C}^3$ we must have

$$3 = \dim \mathbb{C}^3 \geq \dim E_1 \oplus U = \dim E_1 + \dim U = \dim E_1 + 2 \geq 3,$$

since $\dim E_1 \geq 1$. Hence, we have that $\dim E_1 \oplus U = 3$ so that $E_1 \oplus U = \mathbb{C}^3$. Also, since eigenspaces are always B-invariant we have found a B-invariant complement to U.