Math 110, Summer 2012 Short Homework 7

Due Monday 7/12, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Consider the $\mathbb{C}$-vector space $\mathbb{C}_3[t]$ consisting of polynomials with $\mathbb{C}$-coefficients that have degree at most 3. We have $\dim_{\mathbb{C}} \mathbb{C}_3[t] = 4$. Consider the $\mathbb{C}$-linear endomorphism

$$D : \mathbb{C}_3[t] \to \mathbb{C}_3[t] \; ; \; f \mapsto \frac{df}{dt}.$$

a) Show that D is a nilpotent endomorphism and determine the exponent of D, $\eta(D)$.

b) For each k, $0 \leq k \leq \eta(D)$, determine

$$H_k = \{f \in \mathbb{C}_3[t] \mid \mathrm{ht}(f) \leq k\},$$

and determine $\dim H_k = m_k$.

c) Recall the algorithm from Section 2.3 used to determine a basis of V, given a nilpotent endomorphism $g \in \mathrm{End}_{\mathbb{C}}(V)$. Using this algorithm find an ordered basis $\mathcal{B}$ of $\mathbb{C}_3[t]$ such that $[D]_{\mathcal{B}}$ is block diagonal matrix, each block being a 0-Jordan block.

(*Hint: there is only one* 0*-Jordan block.*)

2. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in Mat_2(\mathbb{C}),$$

and consider the endomorphism

$$R_A : Mat_2(\mathbb{C}) \to Mat_2(\mathbb{C}) \; ; \; B \mapsto BA.$$

a) Show that R_A is a nilpotent endomorphism and determine the exponent of R_A, $\eta(R_A)$.

b) For each k, $0 \leq k \leq \eta(R_A)$, determine

$$H_k = \{B \in Mat_2(\mathbb{C}) \mid \mathrm{ht}(B) \leq k\},$$

and determine $\dim H_k = m_k$.

c) As in 1*c*) above, determine an ordered basis $\mathcal{B} \subset Mat_2(\mathbb{C})$ such that $[R_A]_{\mathcal{B}}$ is a block diagonal matrix, each block being a 0-Jordan block.

(*Hint: there is more than one* 0*-Jordan block in this case.*)

Proofs

3. Let $f \in \mathrm{End}_{\mathbb{C}}(V)$, where V is a finite dimensional $\mathbb{C}$-vector space. Denote the eigenvalues of f by $\lambda_1, \dots, \lambda_k$. Prove: f is diagonalisable if and only if, for every i, the algebraic multiplicity of λ_i is equal to the geometric multiplicity of λ_i.

(*Looking at Proposition 2.1.14 and its proof may help here.*)

4. Let $f \in \mathrm{End}_{\mathbb{C}}(V)$, where $\dim V = n$, and suppose that there is an ordered basis $\mathcal{B} = (b_1, \dots, b_n)$ of V such that

$$[f]_{\mathcal{B}} = \begin{bmatrix} A & B \\ 0_{n-k,k} & C \end{bmatrix}.$$

Prove that $U = \mathrm{span}_{\mathbb{C}}\{b_1, \dots, b_k\}$ is f-invariant.