Math 110, Summer 2012 Short Homework 6 (SOME) SOLUTIONS

Due Monday 7/9, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ -2 & -1 & -2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Determine $\chi_A(\lambda)$ and give the eigenvalues of A - there are exactly two distinct eigenvalues, λ_1, λ_2. What is the algebraic multiplicity of each eigenvalue?
Determine a basis of $E_{\lambda_1}, E_{\lambda_2}$, the eigenspaces of A. What is the geometric multiplicity of each eigenvalue? Explain why A is diagonalisable. Give an invertible matrix P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{bmatrix}.$$

Solution: We have

$$\det(A - \lambda I_3) = \chi_A(\lambda) = (1-\lambda)^2(2-\lambda),$$

so that the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$ and with algebraic multiplicity 2 (resp. 1).

By row reducing $A - \lambda_i I_3$, for $i = 1, 2$, we see that

$$E_1 = \text{span}_{\mathbb{C}} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}, \quad E_2 = \text{span}_{\mathbb{C}} \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

And that these spanning sets are linearly independent, hence must form a basis of each eigenspace. We see that the geometric multiplicity of 1 is 2; the geometric multiplicity of 2 is 1. Hence, by a result from class, since the geometric and algebraic multiplicities of each eigenvalue coincides we must have that A is diagonalisable.

If we set

$$P = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & -2 \\ 0 & -1 & 1 \end{bmatrix},$$

then

$$P^{-1}AP = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}.$$

2. Consider the matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in Mat_3(\mathbb{C}).$$

Show that the subspace

$$U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0 \right\} \subset \mathbb{C}^3,$$

is B-invariant and that 1 is an eigenvalue of B. Show that $E_1 \cap U = \{0_{\mathbb{C}^3}\}$. Find a B-invariant subspace $W \subset V$ such that

$$V = W \oplus U.$$

Justify your answer.

Proofs

3. Let V be a finite dimensional $\mathbb{C}$-vector space, $f \in \text{End}_{\mathbb{C}}$. Prove that 0 is an eigenvalue of f if and only if f is not injective.

Solution: Suppose that 0 is an eigenvalue of f. This means that $\ker f = \ker(f - 0.\text{id}_V) \neq \{0_V\}$. Hence, f is not injective.

Conversely, if f is not injective the there is a nonzero vector $v \in \ker f$. Hence, we have that $f(v) = 0_V = 0 \cdot v$, so that $\lambda = 0$ is an eigenvalue of f.

4. Let $A \in Mat_5(\mathbb{C})$. Suppose that $\text{rank}A = 3$ and that A has three distinct nonzero eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{K}$. Prove that A is diagonalisable.

(*You need to use the information given to try and determine what $\chi_A(\lambda)$ looks like so that you can try and use Proposition 2.1.14.*)

Solution: Since $\text{rank}A = 3$ then $\dim \ker T_A = 2$, by the Rank Theorem. Hence, by the previous problem, we see that 0 is an eigenvalue of A with geometric multiplicity 2. Moreover, since there are three distinct nonzero eigenvalues $\lambda_1, \lambda_2, \lambda_3$, each must have geometric multiplicity at least 1.

Hence, since we have

$$E_0 + E_{\lambda_1} + E_{\lambda_2} + E_{\lambda_3} = E_0 \oplus E_{\lambda_1} \oplus E_{\lambda_2} \oplus E_{\lambda_3} \subset \mathbb{C}^5,$$

we see that

$$5 = \dim \mathbb{C}^5 \geq \dim E_0 + \dim E_{\lambda_1} + \dim E_{\lambda_2} + \dim E_{\lambda_3} \geq 2 + 1 + 1 + 1 = 5.$$

Therefore, we must have

$$\mathbb{C}^5 = E_0 \oplus E_{\lambda_1} \oplus E_{\lambda_2} \oplus E_{\lambda_3},$$

so that there exists a basis of $\mathbb{C}^5$ consisting of eigenvectors of A. Hence, A is diagonalisable.

5. Let $f \in \text{End}_{\mathbb{C}}(V)$ and $U \subset V$ an f-invariant subspace. of V. Prove:

- U is also $f^k = f \circ \cdots \circ f$-invariant.
- If U is also g-invariant, for some $g \in \text{End}_{\mathbb{C}}(V)$, then U is $(f + g)$-invariant.
- If $\lambda \in \mathbb{C}$ then U is λf-invariant.
- Prove that $\text{im}f, \ker f$ are f-invariant.

6. Let $f \in \text{End}_{\mathbb{C}}(V)$, with V an n-dimensional $\mathbb{C}$-vector space. Suppose that $f^2 = f \circ f = f$.

- Prove that $V = \text{im}f \oplus \ker f$.
- Prove that the only eigenvalues of f are $\lambda = 0, 1$.

  (*If λ is any eigenvalue, determine a polynomial relation on λ that forces $\lambda = 0, 1$.*)
- Deduce that $\chi_f(\lambda) = \lambda^s(1 - \lambda)^{n-s}$, for some $1 \leq s < n$.
- Prove that $\text{im}f = E_1$ is the 1-eigenspace of f and deduce that $f = p_U$, for $U = \text{im}f$.

  (*Here p_U is the 'projection onto U morphism' discussed on p. 60 of the notes.*)

Solution:

- By the Rank Theorem we see that

$$\dim V = \dim \text{im}f + \dim \ker f.$$

  Hence, if we can show that $\ker f \cap \text{im}f = \{0_V\}$ then we have that $\ker f + \text{im}f = \ker f \oplus \text{im}f$ and

$$\dim \ker f \oplus \text{im}f = \dim \ker f + \dim \text{im}f = \dim V,$$

so that $V = \ker f \oplus \mathrm{im} f$.

Now, let $x \in \ker f \cap \mathrm{im} f$. Then, $x = f(y)$ for some $y \in V$, and

$$0_V = f(x) = f(f(y)) = f(y) = x,$$

where we have used that $f^2 = f$. Hence, $\ker f \cap \mathrm{im} f = \{0_V\}$.

- Let $\lambda \in \mathbb{C}$ be an eigenvalue of f. Then, if v is an eigenvector with associated eigenvalue λ then we have
$$f(v) = \lambda v.$$

  Hence,
$$\lambda v = f(v) = f(f(v)) = f(\lambda v) = \lambda f(v) = \lambda^2 v.$$

  Thus, we must have $(\lambda^2 - \lambda)v = 0_V$, so that $\lambda = \lambda^2$, since $v \neq 0_V$. This can only happen if λ is either 0 or 1.

- Since the only possible eigenvalues of f are $\lambda = 0, 1$ the characteristic polynomial must take the form
$$\chi_f(\lambda) = \lambda^s (1 - \lambda)^{n-s},$$

  since $\deg \chi_f(\lambda) = n$.

- Let $v \in E_1$. Then, $f(v) = v$ so that $v \in \mathrm{im} f$. Conversely, let $x = f(y) \in \mathrm{im} f$; we are going to show that $x \in E_1$. Indeed,
$$f(x) = f(f(y)) = f(y) = x,$$

  so that $x \in E_1$. Hence, we have just shown that $\mathrm{im} f = E_1$.

  In order to deduce that last statement we need to show an equality of functions, that is, we must show that $f(v) = p_U(v)$, for every $v \in V$. Now, since $V = \ker f \oplus \mathrm{im} f$ then we have

$$v = z + u, \ z \in \ker f, \ u \in \mathrm{im} f.$$

  Then,
$$f(v) = f(z + u) = f(z) + f(u) = 0_V + f(u) = 0_V + u,$$

  where we have used that $f(u) = u$, since $\mathrm{im} f = E_1$. Hence, since

$$p_U(v) = u,$$

  we must have that $f(v) = p_U(v)$, for every $v \in V$.