Math 110, Summer 2012 Short Homework 5

Due Thursday 7/5, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Consider the bases
$$\mathcal{S}^{(2)} = (e_1, e_2) \subset \mathbb{Q}^2,\ \mathcal{C} = (f_1, f_2, f_3) \subset \mathbb{Q}^{\{1,2,3\}},$$
where
$$f_1(1) = 1, f_1(2) = 0, f_1(3) = -1;\ f_2(1) = 0, f_2(2) = -1, f_2(3) = -1;\ f_3(1) = 1, f_3(2) = 2, f_3(3) = 0.$$
Consider the following linear morphism
$$\alpha : \mathbb{Q}^2 \to \mathbb{Q}^{\{1,2,3\}}\ ;\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \alpha\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = f : \begin{cases} 1 \mapsto x_1 \\ 2 \mapsto x_2 \\ 3 \mapsto x_1 + x_2 \end{cases}.$$
Determine $[\alpha]^{\mathcal{C}}_{\mathcal{S}^{(2)}}$. Is α injective? Explain your answer. What is $[\alpha]^{\mathcal{B}}_{\mathcal{S}^{(2)}}$, where $B = \{e_i \mid i = 1, 2, 3\} \subset \mathbb{Q}^{\{1,2,3\}}$?

(*To determine $[\alpha]^{\mathcal{C}}_{\mathcal{S}^{(2)}}$ you will need to find the $\mathcal{C}$-coordinates of $\alpha(e_i)$ - to do this it may help to use the change of coordinate matrix $P_{\mathcal{C}\leftarrow\mathcal{B}} = P^{-1}_{\mathcal{B}\leftarrow\mathcal{C}}$.*)

2. Consider the matrix
$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \in Mat_{2,3}(\mathbb{Q}).$$
Determine the rank of A, rank $A = r$, and find matrices $P \in \mathrm{GL}_3(\mathbb{Q})$, $Q \in \mathrm{GL}_2(\mathbb{Q})$ such that
$$Q^{-1}AP = \begin{bmatrix} I_r & 0_{r,3-r} \\ 0_{2-r,r} & 0_{2-r,3-r} \end{bmatrix}.$$
(*Of course, if $r = 2$ then we do not have the bottom row. You need to replicate your proof of Q4.*)

3. Let $\mathcal{S}^{(3)} = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Q}^3$. There are six possible orderings of $\mathcal{S}^{(3)}$: write them all down to obtain six different ordered bases $\mathcal{B}_1, \dots, \mathcal{B}_6$ and so that $\mathcal{B}_1 = (e_1, e_2, e_3)$. Write down the change of coordinates matrices $P_{\mathcal{B}_i\leftarrow\mathcal{B}_1}$, for $i = 1, \dots, 6$.

Proofs

4. Let $f, g \in \mathrm{Hom}_{\mathbb{K}}(V, W)$, $\mathcal{B} = \{b_1, \dots, b_n\} \subset V$ a basis of V. Prove that if $f(b_i) = g(b_i)$, for each $i = 1, \dots, n$, then $f = g$.

(*In order to show that two functions $f, g : V \to W$ are equal, you must show that $f(v) = g(v)$, for every $v \in V$. Therefore, this questions tells us that in order to show two linear morphisms are equal, it suffices to check that they are equal on a basis.*)

5. Let $A \in Mat_{m,n}(\mathbb{K})$ be such that rank $A = r$. Prove that there exists $P \in \mathrm{GL}_n(\mathbb{K}), Q \in \mathrm{GL}_m(\mathbb{K})$ such that
$$Q^{-1}AP = \begin{bmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}.$$
(*Replicate the proof of Theorem 1.7.14 for $f = T_A$. How do P, Q arise?*)