Math 110, Summer 2012 Short Homework 4

Due Monday 7/2, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Which of the following subsets are bases of the vector space V? Explain your answer.

$$A = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^3,$$

$$B = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ -2 & 2 \end{bmatrix} \right\} \subset Mat_2(\mathbb{Q}),$$

$$C = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} \subset U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0 \right\}.$$

2. Consider the linear morphism

$$\text{tr} : Mat_2(\mathbb{R}) \to \mathbb{R} \; ; \; A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto a_{11} + a_{12}.$$

Determine an ordered basis $\mathcal{B}$ of the subspace $U = \ker \text{tr} \subset Mat_2(\mathbb{R})$, and an ordered basis $\mathcal{C} \subset \mathbb{R}$ of $\mathbb{R}$ making sure to explain why you know that the ordered sets you give are bases.

Using the ordered bases $\mathcal{B}$ and $\mathcal{C}$ you have found, determine the matrix $[\text{tr}]_{\mathcal{B}}^{\mathcal{C}}$ of tr relative to $\mathcal{B}$ and $\mathbb{C}$.

3. Consider the vector subspace (you DO NOT have to show this)

$$S_n = \{A \in Mat_n(\mathbb{Q}) \mid A = A^t\} \subset Mat_n(\mathbb{Q}),$$

where A^t is the transpose of A (so that if $A = [a_{ij}]$ then $A^t = [a_{ji}]$). S_n consists of all *symmetric* $n \times n$ *matrices with* $\mathbb{Q}$*-entries*.

a) Determine a basis $\mathcal{B}$ of S_n and show that the subset you obtain is a basis.

b) Find a closed formula for the dimension of S_n.

(*It might help to consider what happens when* $n = 2, 3, 4$ *first*)

Consider the subspace

$$A_n = \{A \in Mat_n(\mathbb{Q}) \mid A = -A^t\} \subset Mat_n(\mathbb{Q}).$$

A_n consists of all *antisymmetric* $n \times n$ *matrices eith* $\mathbb{Q}$*-entries*.

c) Determine a basis $\mathcal{C}$ of A_n and show that the subset you obtain is a basis.

d) Find a closed formula for the dimension of A_n.

e) Show that $S_n \cap A_n = \{0_n\}$ and deduce that $Mat_n(\mathbb{Q}) = A_n \oplus S_n$.

f) You have just shown that $\mathcal{D} = \mathcal{B} \cup \mathcal{C}$ is a basis of $Mat_n(\mathbb{Q})$. Find the $\mathcal{D}$-coordinates of the matrix

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 2 \\ -1 & -2 & 1 \end{bmatrix} \in Mat_3(\mathbb{Q}).$$

Proofs

4. Let V be a $\mathbb{K}$-vector space, $\mathcal{B} = (b_1, \dots, b_n)$. Prove that

$$V = \bigoplus_{i=1}^{n} \text{span}_{\mathbb{K}}\{b_i\} = \text{span}_{\mathbb{K}}\{b_1\} \oplus \cdots \oplus \text{span}_{\mathbb{K}}\{b_n\}.$$

(*You must show that* $V = \text{span}_{\mathbb{K}}\{b_1\} + \dots + \text{span}_{\mathbb{K}}\{b_n\}$ *and that this sum is direct*)