Math 110, Summer 2012 Short Homework 3

Due Monday 6/27, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Is the function

$$\alpha : \mathbb{Q}[t] \to \mathbb{Q}[t] \; ; \; f \mapsto t^3 f - 3t,$$

a $\mathbb{Q}$-linear morphism? Justify your answer. Here $\mathbb{Q}[t]$ is the $\mathbb{Q}$-vector space of polynomials defined in the notes.

2. Which of the following functions are $\mathbb{K}$-linear? Justify your answers.

$$f : \mathbb{R}^2 \mapsto \mathbb{R}^4 \; ; \; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 3x_1 + 2x_2 \\ \exp(x_1) \\ 0 \\ -x_1 \end{bmatrix}, \; (\mathbb{K} = \mathbb{R})$$

$$g : Mat_{3,2}(\mathbb{C}) \mapsto Mat_{2,3}(\mathbb{C}) \; ; \; A \mapsto PAQ, \text{ where } P, Q \in Mat_{2,3}(\mathbb{Q}) \text{ are fixed}, \; (\mathbb{K} = \mathbb{C})$$

$$h : \mathbb{Q}^{\{1,2,3\}} \mapsto Mat_{2,2}(\mathbb{Q}) \; ; \; (f : i \mapsto f(i)) \mapsto \begin{bmatrix} f(1) & 2f(2) + 3f(3) \\ 0 & -f(1) \end{bmatrix}. \; (\mathbb{K} = \mathbb{Q})$$

Proofs

3. Let P be the set of positive numbers, so $P = (0, \infty)$. Define

$$\alpha : P \times P \to P \; ; \; (x, y) \mapsto xy, \quad \sigma : \mathbb{R} \times P \to P \; ; \; (\lambda, x) \mapsto x^\lambda.$$

Show that (P, α, σ) is an $\mathbb{R}$-vector space. You must check Axioms VS1-VS8 and you need to define $0_P \in P$ and, for any $x \in P$, $-x \in P$.

Can you explain how this 'weird' $\mathbb{R}$-vector space arises? (*Hint: there is a bijective function $L : P \to \mathbb{R}$ that might help you understand why we have defined 'addition' as 'multiplication'.*)

4. Let V be a $\mathbb{K}$-vector space, $E \subset V$ a nonempty subset. Prove that $\text{span}_{\mathbb{K}} E$ is equal to the intersection of all subspaces $U \subset V$ such that $E \subset U$. So, if $\mathcal{F}$ is the set of all subspaces of V that contain E (ie, $U \in \mathcal{F}$ if and only if $E \subset U$), then prove that

$$\text{span}_{\mathbb{K}} E = \bigcap_{U \in \mathcal{F}} U.$$

(*Hint: to show that two sets A, B are equal, it suffices to show that $A \subset B$ and $B \subset A$.*)

5. Let V, W be $\mathbb{K}$-vector spaces, $f \in \text{Hom}_{\mathbb{K}}(V, W)$ an isomorphism. Let E be a nonempty subset of V. Prove:

- E is linearly independent in V if and only if $f(E)$ is linearly independent in W.
- E spans V if and only if $f(E)$ spans W.

Here, we define $f(E) = \{f(e) \mid e \in E\}$.