Math 110, Summer 2012 Short Homework 2

Due Monday 6/25, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Determine the linear (in)dependence of the following subsets:

$$E_1 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{Q}^3,$$

$$E_2 = \left\{ I_2, A, A^2 \right\} \subset M_2(\mathbb{R}), \quad \text{where } A = \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{bmatrix}, \ I_2 \text{ is the } 2 \times 2 \text{ identity matrix.}$$

2. Find a vector $v \in E$ such that $\text{span}_{\mathbb{K}} E = \text{span}_{\mathbb{K}} E'$, where

$$E = \left\{ I_2, B, B^2, B^3 \right\} \subset Mat_2(\mathbb{Q}), \quad \text{where } B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

and $E' = E \setminus \{v\}$.

3. Let $V = \mathbb{R}^3$. Consider two planes $\Pi_1, \Pi_2 \subset \mathbb{R}^3$ that pass through the origin. Consider the corresponding vector subspaces $U_1, U_2 \subset \mathbb{R}^3$. Under what conditions must we have $U_1 + U_2 = \mathbb{R}^3$? Is it possible for $U_1 \cap U_2 = \{0_{\mathbb{R}^3}\}$?

Suppose that $W = \text{span}_{\mathbb{K}}\{v\} \subset \mathbb{R}^3$, for $v \in \mathbb{R}^3$. Under what conditions can we have $U_1 + W = \mathbb{R}^3$? Is it possible for $\mathbb{R}^3 = U_1 \oplus W$? Explain your answer.

Proofs

4. Let V be a $\mathbb{K}$-vector space, $U, W \subset V$ vector subspaces of V. Prove:

- $U + W$ is a vector subspace of V,
- $U \cap W$ is a vector subspace of V,
- $U \cup W$ is a vector subspace if and only if $U \subset W$ or $W \subset U$.

Give an example of two subspaces of $U, W \subset \mathbb{R}^2$ such that $U \cup W$ is not a subspace of $\mathbb{R}^2$.

5. Let V be a $\mathbb{K}$-vector space and $A, B \subset V$ be nonempty subsets of V. Prove:

$$\text{span}_{\mathbb{K}}(A \cup B) = \text{span}_{\mathbb{K}} A + \text{span}_{\mathbb{K}} B.$$

6. Let $f \in \text{Hom}_{\mathbb{K}}(V, W)$, where V, W are $\mathbb{K}$-vector spaces. Prove:

- $\ker f \subset V$ is a vector subspace of V,
- $\text{im} f \subset W$ is a vector subspace of W.