Math 110, Summer 2012 Short Homework 1, (SOME) SOLUTIONS

Due Wednesday 6/20, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Some warm-up calculations

1. Row-reduce the following matrices to reduced echelon form:

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Solution: You should find that

$$A \sim I_3, \quad B \sim \begin{bmatrix} 1 & 0 & 0 & -1/7 \\ 0 & 1 & 0 & -3/7 \\ 0 & 0 & 1 & 12/7 \end{bmatrix}.$$

2. For the following matrix equations determine whether a solution exists. If so, determine all possible solutions.

$$A\underline{x} = \underline{0}, \quad B\underline{x} = \underline{0}, \quad A\underline{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Solution: For a homogeneous matrix equation a solution always exists. hence, $A\underline{x} = \underline{0}$ and $B\underline{x} = \underline{0}$ always have solutions (namely the solution $\underline{x} = \underline{0}$).

Since $A \sim I_3$ then $A\underline{x} = \underline{0}$ if and only if $I_3\underline{x} = \underline{0}$. Hence, there is only one solution to $A\underline{x} = \underline{0}$, namely the trivial solution.

We have

$$B\underline{x} = \underline{0} \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & -1/7 \\ 0 & 1 & 0 & -3/7 \\ 0 & 0 & 1 & 12/7 \end{bmatrix} \underline{x} = \underline{0}.$$

Hence, we must have $\underline{x} = \begin{bmatrix} x/7 \\ 3x/7 \\ -12x/7 \\ x \end{bmatrix}$, for any $x \in \mathbb{K}$.

Since A has a pivot in every row then, for any $\underline{b}$, $A\underline{x} = \underline{b}$ has a solution: if you form the augmented matrix $[A \mid \underline{b}]$ and row-reduce you will obtain the reduced echelon form $[I_3 \mid \underline{c}]$. Then, $\underline{c}$ is a solution. In fact, it's the unique solution.

3. Without using determinants show that A is invertible.

Solution: Since A has a pivot in every row/column then A is an invertible matrix **OR** since the columns of A are linearly independent then A is invertible **OR** etc.

Problems from the notes

4. Show that $\mathbb{Q}$ is not a vector subspace of the $\mathbb{R}$-vector space $\mathbb{R}$ (where we have the 'usual' notions of addition and scalar multiplication).

Solution: We have $1 \in \mathbb{Q}$ but $\sqrt{2} \cdot 1 \notin \mathbb{Q}$, so that $\mathbb{Q}$ is not closed under scalar multiplication by $\mathbb{R}$-scalars.

5. Prove that, if (V, α, σ) is a $\mathbb{K}$-vector space, and $\lambda \in \mathbb{K}$, then $\sigma(\lambda, 0_V) = 0_V$. (*In your solution you can write* $\lambda \cdot 0_v$ *instead of* $\sigma(\lambda, 0_V)$)

Solution: Let $\lambda \in \mathbb{K}$. Then,

$$\begin{aligned} \lambda 0_V &= \lambda(0_V + 0_V), \quad \text{by VS3}, \\ &= \lambda 0_V + \lambda 0_V, \quad \text{by VS7}, \\ \implies \quad 0_V &= \lambda 0_V, \quad \text{by subtracting } 0_V \text{ from both sides.} \end{aligned}$$

6. Prove that V and $\{0_V\}$ are subspaces of a $\mathbb{K}$-vector space V.

7. Prove: if a nonempty subset $U \subset V$ satisfies Axiom SUB then U satisfies Axiom SUB1, SUB2 and SUB3.

Solution: Suppose that U satisfies SUB.

SUB1 take $u = v = 0_V$, $\lambda = \mu = 1$, then $0_V = 1.u + 1.v \in U$.

SUB2 take $u, v \in V$, $\lambda = \mu = 1$, then $u + v = 1.u + 1.v \in U$.

SUB3 take $u, v \in V$, $\lambda \in \mathbb{K}, \mu =$, then $\lambda u = \lambda u + 0.v \in U$.