Math 110, Summer 2012 Short Homework 11 (SOME) SOLUTIONS

Due Thursday 8/2, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

0. Was this homework assignment too easy/too difficult/about right? Any other comments are welcome.

Calculations

1. Show that the following bilinear form is an inner product on $\mathbb{R}^3$,

$$B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \; ; \; (\underline{x}, \underline{y}) \mapsto x_1y_1 + 2x_2y_2 + 3x_3y_3 - x_1y_2 - x_2y_1 - x_2y_3 - x_3y_2.$$

(*You must show that B is symmetric, nondegenerate and positive definite.*)

Determine an Euclidean isomorphism

$$f : (\mathbb{R}^3, B) \to \mathbb{E}^3.$$

What is the length of $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, with respect to B?

Solution: Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

Then, $B = B_A$: indeed, for any $\underline{x}, \underline{y} \in \mathbb{R}^3$ we have

$$B_A(\underline{x}, \underline{y}) = \underline{x}^t A \underline{y} = x_1y_1 + 2x_2y_2 + 3x_3y_3 - x_1y_2 - x_2y_1 - x_2y_3 - x_3y_2.$$

Since A is symmetric and invertible then B is symmetric and nondegenerate. Also, we have

$$B(\underline{x}, \underline{x}) = x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 = (x_1 - x_2)^2 + (x_2 - x_3)^2 + 2x_3^2 \geq 0, \text{ for every } \underline{x} \in \mathbb{R}^3,$$

and

$$B(\underline{x}, \underline{x}) = 0 \Leftrightarrow x_3 = 0,\; x_2 - x_3 = 0,\; x_1 - x_2 = 0 \Leftrightarrow x_1 = x_2 = x_3 = 0.$$

Consider the coordinates

$$y_1 = x_1 - x_2,\; y_2 = x_2 - x_3,\; y_3 = \sqrt{2}x_3,$$

so that

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \; (= Q\underline{x}).$$

Then, if we let

$$P = Q^{-1} = \begin{bmatrix} 1 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix},$$

we have that $P^tAP = I_3$. Now,

$$T_Q : \mathbb{R}^3 \to \mathbb{R}^3 \; ; \; \underline{x} \mapsto Q\underline{x},$$

is an isomorphism of $\mathbb{R}$-vector spaces and

$$B_A(\underline{x}, \underline{y}) = \underline{x}^t A \underline{y} = \underline{x}^t Q^t Q \underline{y} = (Q\underline{x})^t (Q\underline{y}) = (Q\underline{x}) \cdot (Q\underline{y}),$$

so that $f = T_Q$ is a Euclidean isomorphism.
Recall that $||\underline{x}|| = \sqrt{B(\underline{x}, \underline{x})}$, so we have

$$\left\| \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\| = \sqrt{B\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right)} = \sqrt{4 + 9 + 8} = \sqrt{21}.$$

2. Using the inner product B above, determine the orthogonal complement of $\text{span}_{\mathbb{R}}\{e_1\} \subset \mathbb{R}^3$ (with respect to B).

Solution: We have

$$\text{span}_{\mathbb{R}}\{e_1\}^{\perp} = \{e_1\}^{\perp} = \{v \in \mathbb{R}^3 \mid B(v, e_1) = 0\}.$$

Let $\underline{x} \in \mathbb{R}^3$. Then, we want that

$$0 = B(\underline{x}, e_1) = x_1 - x_2 \implies x_1 = x_2.$$

Hence, we have

$$\text{span}_{\mathbb{R}}\{e_1\}^{\perp} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 = x_2 \right\}.$$

3. Show that the following bilinear form is NOT and inner product

$$B' : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \; ; \; (\underline{x}, \underline{y}) \mapsto x_1 y_1 + x_2 y_3 + x_3 y_2,$$

by finding a vector $\underline{x}_0 \in \mathbb{R}^3$ such that $B'(\underline{x}_0, \underline{x}_0) < 0$.

(*Hint: determine the canonical form of B'.*)

Solution: The canonical form of B' is

$$B'(\underline{x}, \underline{x}) = x_1^2 + 2x_2x_3 = x_1^2 + \frac{1}{2}((x_2 + x_3)^2 - (x_2 - x_3)^2),$$

so if we take

$$\underline{x}_0 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

then we have $B'(\underline{x}_0, \underline{x}_0) = -2 < 0$.

Proofs

4. Prove Pythagoras' theorem (Theorem 3.3.6).

Solution: Suppose that $u, v \in V$ such that $\langle u, v \rangle = 0$. Then,

$$||u + v||^2 = \langle u + v, u + v \rangle = \langle u, u \rangle + \langle v, v \rangle + \langle u, v \rangle + \langle v, u \rangle = ||u||^2 + ||v||^2 + 0 + 0 = ||u||^2 + ||v||^2,$$

where we have used that $\langle v, u \rangle = \langle u, v \rangle = 0$.

5. Prove the Cauchy-Schwarz inequality (Theorem 3.3.6) as follows: let $u, v \in V$.

- if $v = 0_V$ then the result is easy (you must still show this!).
- if $v \neq 0_V$ then consider

  $$\langle u - \lambda v, u - \lambda v \rangle, \text{ for any } \lambda \in \mathbb{R}.$$

  By making an informed choice of λ (expand out the above expression) you will obtain

  $$\langle u, u \rangle \langle v, v \rangle \geq \langle u, v \rangle^2.$$

  Use this to deduce the result.

Solution: Suppose that $u, v \in V$, $v = 0_V$. Then,

$$0 = \langle u, v \rangle \leq ||u|| ||v|| = 0.$$

Now, suppose that $v \neq 0_V$. Then, for any $\lambda \in \mathbb{R}$,

$$0 \leq \langle u - \lambda v, u - \lambda v \rangle = \langle u, u \rangle + \lambda^2 \langle v, v \rangle - 2\lambda \langle u, v \rangle.$$

Let

$$\lambda = \frac{\langle u, v\rangle}{\langle v, v\rangle} \in \mathbb{R},$$

(this is well-defined as $v \neq 0_V$) then the above expression gives

$$0 \leq \langle u, u\rangle + \frac{\langle v, u\rangle^2}{\langle v, v\rangle} - 2\frac{\langle v, u\rangle^2}{\langle v, v\rangle} \implies \langle v, u\rangle^2 \leq \langle u, u\rangle\langle v, v\rangle,$$

since $\langle v, v\rangle > 0$. Hence, taking the (positive) square root of both sides gives the result.

6. Prove that an Euclidean morphism $f : (V_1, \langle,\rangle_1) \to (V_2, \langle,\rangle_2)$ is injective.

Solution: Let $f : V_1 \to V_2$ be an Euclidean morphism. Then, for any $v \in \ker f$, we have

$$0 = \langle f(v), f(v)\rangle_2 = \langle v, v\rangle_1 \implies ||v||_1^2 = 0 \implies ||v||_1 = 0 \implies v = 0_V.$$

7. Let $(V, \langle,\rangle)$ be an Euclidean space, $S \subset V$ a nonempty subset. Prove that $S^\perp$ is a subspace and that

$$(\text{span}_\mathbb{R} S)^\perp = S^\perp.$$

(*To show two sets A, B are equal, it suffices to show that $A \subset B$ and $B \subset A$.*)

Solution: Let $u, v \in S^\perp$, $\lambda, \mu \in \mathbb{R}$. Then, for any $s \in S$ we have

$$\langle s, \lambda u + \mu v\rangle = \lambda\langle s, u\rangle + \mu\langle s, v\rangle = 0.$$

Hence, $\lambda u + \mu v \in S^\perp$.

Let $x \in (\text{span}_\mathbb{R} S)^\perp$, so that $\langle x, u\rangle = 0$, for every $u \in \text{span}_\mathbb{R} S$. In particular, for every $s \in S \subset \text{span}_\mathbb{R} S$ we have $\langle x, s\rangle = 0$. Hence, $x \in S^\perp$.

Conversely, suppose that $x \in S^\perp$ and let $u = c_1 s_1 + ... + c_k s_k \in \text{span}_\mathbb{R} S$. Then,

$$\langle x, u\rangle = \langle x, c_1 s_1 + ... + c_k s_k\rangle = c_1\langle x, s_1\rangle + ... + c_k\langle x, s_k\rangle = 0 + ... + 0 = 0.$$

Hence, $x \in (\text{span}_\mathbb{R} S)^\perp$ and the result follows.