Math 110, Summer 2012 Long Homework 6

Due Wednesday 8/8, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Please write your answers in complete English sentences (where applicable). Make your arguments rigorous - if something is 'obvious', state why this is the case. Full credit will be awarded to those solutions that are complete and answer the question posed in a coherent manner.

1. In this problem we will see that every orthogonal matrix $A \in O(2)$ is of the form

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} R_\theta, \text{ or } A = R_\theta,$$

where

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \ \theta \in [0, 2\pi).$$

a) Prove that if $A \in O(2)$ then $\det A = \pm 1$.

b) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in O(2).$$

Using the fact that $A^{-1} = A^t$, show that

$$(a, b) = (d, -c) \text{ if } \det(A) = 1, \text{ and } (a, b) = (-d, c) \text{ if } \det(A) = -1.$$

c) Using *b*) show that

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \text{ or } A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix},$$

where $a, b \in \mathbb{R}$ are such that $a^2 + b^2 = 1$. Deduce that there exists unique $\theta \in [0, 2\pi)$ such that

$$a = \cos\theta, \ b = \sin\theta.$$

d) Prove that either

$$A = R_\theta, \text{ or } A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} R_\theta.$$

2. In this problem we will show that if $A \in O(3)$ is such that $\det A = 1$, then there exists $P \in O(3)$ such that

$$P^t A P = \begin{bmatrix} R_\theta & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, any orthogonal transformation with determinant 1 corresponds to 'rotation about a line L in $\mathbb{R}^3$'.

Let $A \in O(3)$ be such that $\det A = 1$. Denote the eigenvalues of A, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ (recall that it may not be possible that all eigenvalues are <u>real</u>: for example, the matrix

$$Z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in O(3)).$$

All notions of length, orthogonality in this problem will be with respect to the 'dot product'.

a) Show that A has at least one real eigenvalue. (*Hint: every cubic equation admits at least one real root.*)

b) Let λ be a real eigenvalue of A. Show that $|\lambda| = 1$. Deduce, that $\lambda = \pm 1$. (*Hint: What properties do orthogonal transformations (=Euclidean isomorphisms) satisfy?*)

c) Recall that

$$A^t A = I_3 = AA^t,$$

so that A is **normal**. Using the result on eigenspaces of normal morphisms, show that if there is $P \in \mathrm{GL}_3(\mathbb{R})$ such that $P^{-1}AP = D$, where $D \in Mat_3(\mathbb{R})$ is diagonal, then there exists $Q \in O(3)$ such that $Q^t AQ = D$. (*Hint: Gram-Schmidt process*).

d) Suppose that A is diagonalisable, so that there exists $P \in \mathrm{GL}_3(\mathbb{R})$ such that $P^{-1}AP = D$, with $D \in Mat_3(\mathbb{R})$ diagonal. Using b) show that the entries $d_1, d_2, d_3 \in \mathbb{R}$ (ie the eigenvalues of A) on the diagonal of D are such that $d_1, d_2, d_3 \in \{1, -1\}$ and that the number of 1s appearing on the diagonal is odd. (*Hint: Use that* $\lambda_1\lambda_2\lambda_3 = \det A = 1$.)

e) Deduce that when $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ then there exists $Q \in O(3)$ such that either

$$Q^t AQ = I_3, \quad \text{or} \quad Q^t AQ = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}.$$

(*Hint: A is normal, hence diagonalisable.*)

f) **For the remaining problems assume that not all of the eigenvalues of A are real (so that $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$, for some i)**. Prove that there are precisely two non-real eigenvalues. (*Hint: Use that* $\lambda_1\lambda_2\lambda_3 = \det A = 1$ *and a)*)

g) Denote the real eigenvalue $\lambda_1 \in \mathbb{R}$, so that $\lambda_2, \lambda_3 \in \mathbb{C} \setminus \mathbb{R}$ (by f)). Let $E_1 = E_{\lambda_1}$ denote the λ_1-eigenspace of A. Show that $\dim E_1 = 1$ and that $E_1^{\perp}$ is A-invariant.

h) Using g) and the fact that A is normal show that there is $Q \in O(3)$ such that

$$Q^t AQ = \begin{bmatrix} R & 0 \\ 0 & \lambda_1 \end{bmatrix},$$

where $R \in O(2)$. (*Hint: Use properties of eigenspaces of normal morphisms and Gram-Schmidt.*)

i) Show that if $S \in O(2)$ is such that $\det S = -1$, then A is similar to $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Deduce that in h) we must have $\det R = 1$ and $\lambda_1 = 1$. (*Hint: if* $X = \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix}$ *then the eigenvalues of X are the eigenvalues of Y together with the eigenvalues of Z.*)

j) Deduce that if $A \in O(3)$ with $\det A = 1$ then there exists $Q \in O(3)$ such that

$$Q^t AQ = \begin{bmatrix} R_\theta & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{for some } \theta \in [0, 2\pi).$$